Consumption and Labor-Leisure Choices: YOLO and Long-Run Sustainability*

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Abstract

This paper investigates factors causing different lifestyle choices and their long-run implications in a rational economic agent framework. To accomplish this, we establish a dynamic model of consumption, *labor-leisure* allocation, and risky investment decisions for an agent with recursive preference. We first characterize four different lifestyles including YOLO (You Only Live Once) based on the agent's consumption and labor-leisure patterns, and then explore the long-term sustainability of these lifestyles. We discover that lifestyle choices are time-varying and can dramatically change according to the agent's financial status. These findings provide profound policy implications.

JEL classification: C61, D11, D15, G11, G51, J22

Keyword: YOLO, labor-leisure choice, sustainability, recursive utility, free boundary value problems

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1 Introduction

With the rise of the YOLO (You Only Live Once!) mindset since the 2010s, an increasing trend, particularly among the young, is placing a higher emphasis on embracing leisure in the early stages of life. In contrast, over the last two decades, there has been a significant surge in the proportion of individuals engaging in multiple job commitments.[1] Indeed more flexibility in working hour choices enables people to pursuit diverse lifestyles. This trend has been significantly facilitated by the emergence of online platforms, exemplified by companies like Uber and Lyft (Campbell (2018) and Hall and Krueger (2018)). There is significant heterogeneity in work schedules among the working populations and notably, the heterogeneity is not caused by the difference in productivity, job characteristics, or wages, but by workers' preference (Lachowska et al., 2023).

Given the increasing flexibility in working hours for individuals, many mass media outlets have raised serious concerns about lifestyles of the young generations.[2] Critiques often point out that they tend to travel extensively, work few hours, prioritize consumerism, and show little concern for the potential long-term consequences of their lifestyles. In fact, when viewed collectively, these lifestyles can pose significant challenges on the social security, insurance, and health care systems when the current young reaches old age. This is because these choices often do not lead to sufficient savings for their later years. It is also notable according to the 2018 report from the National Institute on Retirement Security that 66% of U.S. working millennials have no employer-provided savings retirement (Brown, 2018). These trends raise important questions about the financial well-being of current young (and future old) generations and the sustainability of support systems in the long-run.

Why do some people work more or less?[3] Specifically, what preference factors lead to life patterns such as YOLO or non-YOLO? Is YOLO sustainable? That is, can an individual who takes a YOLO lifestyle when young still keep the same pattern in later life? Most

[1] According to data from the U.S. Census Bureau's Longitudinal Employer-Household Dynamics (LEHD), the percentage of employed individuals holding more than one job has risen to 7.2% during the period spanning from the second quarter of 1996 to the first quarter of 2018.

[2] For example, see the Forbes article: "The YOLO Mistake Many Millennials Are Making" `https://tinyurl.com/msap2tpu`.

[3] We do not consider the case of involuntary layoffs in our model. If we introduce the layoff shock such as a Poisson process, it will lead to an increase in the subjective discounting factor, which does not fundamentally change the main result of the model.

importantly, what is the reason that YOLO is particularly popular among millenials and Gen Z, while it was not prevalent when the older generations (Baby Boomers and Gen X) were young? Are there any long-term policy implications to mitigate concerns about the new lifestyles of the younger generation? This paper aims to address these questions within a *rational* agent framework.

To achieve this goal, we set up a dynamic optimal consumption, labor-leisure choice, and risky portfolio decision problem of an economic agent with recursive (Epstein-Zin) utility. It is worth highlighting two pivotal elements in our modeling approach that differentiate it from conventional consumption-investment models: (i) *flexible labor-leisure choice* and (ii) *recursive utility*. Our model is the first one to have both (i) and (ii) with closed form solutions in a continuous-time setup while there have been studies that have only either one or the other. The first component is a prerequisite for modeling the heterogeneity or diversity of working hour choices. The second component, i.e., the Epstein-Zin utility setup,[4] is also important because the model implications under the recursive utility setup are fairly different from those under the time-separable utility setup.

	low EIS	high EIS
high ρ (impatient)	YOLO (Type Y)	extreme YOLO (Type Y_e)
low ρ (patient)	Non-YOLO (Type N)	extreme Non-YOLO (Type N_e)

Table 1: Types of agents according to the agent's EIS (ψ) and subject discounting (ρ). See Section 4 for more details. See Figure 10 for a graphical representation in a $(\psi, \rho)-$plain.

As presented in Table 1, we classify economic agents into four distinct types based on the relative levels of the two preference factors: EIS and subjective discounting. First, the extent of (im)patience can serve as a distinguishing factor between YOLO and non-YOLO behaviors as high discounting is indicative of increased tendencies towards both current consumption and leisure. Second and more important, the impact of EIS is opposite for the two cases with high and low discounting in the characterization in Table 1, which is a novel finding in the literature regarding the role of EIS. When the agent is patient (or the financial market is attractive), the substitution effect becomes stronger than the income effect as EIS increases. That is, as EIS increases, the agent is more willing to substitute current leisure and

[4] See Epstein and Zin (1989); Weil (1990), and Duffie and Epstein (1992a,b).

consumption with leisure and consumption in the future. Thus, the current working hours increase with EIS. However, when the agent is impatient, the opposite happens: the income effect becomes stronger than the substitution effect as EIS increases. Thus, as EIS increases, the agent increases his/her current leisure and consumption: the working hours decrease with EIS. In summary, ceteris paribus, the order in the total working hours is Y_e, Y, N, and N_e from lowest to highest.

Based on the above categorization, we provide the long-run implications over the lifetime; It is particularly important to note that EIS plays a pivotal role in determining the sustainability. We show by simulations that YOLO is sustainable for individuals who are impatient and have low EIS (Type Y) in that their total working hours are steady in the long-run. However, YOLO can lead to a disappointing outcome for individuals who are impatient and have high EIS (Type Y_e). They need to work significantly more when they are old than they did when they were young, which is against the spirit of YOLO, since on average they become poorer and poorer due to less work in their early lives. On the other hand, for Type N and N_e individuals, they work much longer early in life, especially for N_e, but their working hours continually decrease over time on average as a result of wealth accumulation.

There are two important notes. First, the sustainability of a life pattern is a matter of the agent's working hour choice: consumption choice for any type of agent has consistency in that the ratio of consumption to the minimum human wealth is fairly stable over time. Second and more important, our result highlights the important role of EIS in the sophisticated characterization of life patterns. Note in the special case of the standard additive-separable utility setup, life styles of impatient agents are always unsustainable; however, in the recursive utility case, we find that even if an agent is fairly impatient, there exists a cutoff of EIS level below which the agent's life style becomes sustainable YOLO (Type Y).

What are policy implications? Before we delve into them, let us start with several related questions: if a lifestyle is solely governed by the preference, what do we learn from this exercise? Or is the reason why millenials and Gen Z have different consumption and leisure patterns from the previous generation is that they were simply born with different preference? The answer is 'NO'. There is no empirical evidence that current young generations are inherited with different preference from old generations.[5] While we categorize the lifestyles by the preference so far, there is another important driver: financial wealth. More precisely, consider the pair of EIS and time preference: (ψ, ρ). Then, there is a region of (ψ, ρ)'s that corresponds to Y_e as in Table 1 or more precisely in Figure 10. We find that this region decreases with financial wealth. That is, types are time-varying and the probability of an individual being Type Y_e becomes higher as wealth decreases. Note that a high EIS agent can be either Type N_e or Type Y_e depending on wealth. This fact provides a striking implication that not only a Type Y agent but also even an extreme Non-YOLO type agent (N_e) can switch to choose a completely different life pattern, i.e., extreme YOLO type (Y_e) if the agent gets hit by a significantly bad shock (see Figure 11). In other words, a large financial setback can lead to a dramatic reduction in working hours for many individuals.

It is crucial to acknowledge that the financial status of the young generations is much weaker than that of the previous generations. In particular, when comparing the periods during which each generation entered the workforce after graduation, it is noteworthy that the millenials and Gen Z face daunting hurdles: high student loan debts, meager retirement savings, soaring housing prices and costs, and lower lifetime incomes (see, e.g., Brown (2018) and (FINESSE, 2017)). Given these factors weakening financial status, our analysis indicates that the current young generation is more likely to *optimally* choose extreme and unsustainable YOLO lifestyles compared to previous generations.

In light of our findings, policy interventions aimed at improving the financial standing of young generations can play a pivotal role in encouraging a shift toward more sustainable lifestyles. For instance, substantial relief from student loans could be considered. The government can provide tax credits for firms that offer more retirement savings for employees. Additionally, they can consider tax credits or subsidies to facilitate home ownership among this age group. The point is that addressing these financial challenges can incentivize young individuals to work more, accumulate higher wealth, and transition to more sustainable life choices, which can eventually help the sustainability of the health care and social security systems in the long-run.

[5] To our best knowledge, there is no such literature in economics. The organization, psychology, and ethic literatures do not find any significance generational differences in work ethic. For example, see Weeks and Schaffert (2019), Zabel et al. (2017), and Pyoria et al. (2017).

Lastly, one may wonder how the agent's risky investment contributes to the wealth accumulation and whether it has long-run implications. It is well known that the optimal risk asset holdings do not depend on the agent's EIS if the investment opportunity is constant (Svensson (1989) and Bhamra and Uppal (2006)). This result holds under a general environment (e.g., a regime switching investment opportunity and the existence of transaction costs and taxes: Cai et al. (2018)). We find that the risky asset holdings do depend on the agent's EIS if the labor-leisure choice is flexible with the minimum working hour requirement. However, we also find that its quantitative effect is limited. That is, the difference in the ratio of the risky asset holdings to the minimum human wealth among different types of agents is negligible. This quantitative result suggests an important implication: The heterogeneity of wealth accumulation among the different types of agents is driven by their consumption and labor-leisure choices over time, not by the result of their risk-taking behaviors.

Literature Review: This paper is closely related to the growing literature on sustainability of consumption, as explored by Arrow et al. (2004), Campbell and Martin (2022), and Pindyck (2022). In these models, sustainability implies that a representative agent with a CRRA (constant relative risk aversion) utility function self-imposes the constraint that its utility value does not decrease over time. Arrow et al. (2004) considers this in the deterministic sense, while Campbell and Martin (2022) and Pindyck (2022) discuss it in the expectation sense. These models focus on aggregate consumption to ensure that future generations are as well off as the current generation. In our model, sustainability centers around individual behavior: specifically, whether an individual, when old, will be able to maintain the same leisure pattern they enjoyed in their youth. Unlike these models, where sustainability breaks down if the rate of time preference is excessively high due to the use of the CRRA utility function, we demonstrate that the consumption and leisure choices of agents with high subjective discounting remain sustainable when the agent's EIS is low. Our most significant finding is that lifestyle choices vary over time based on both wealth and preference. Leveraging these results, we offer policy suggestions on how to address the potential long-term challenges posed by the unsustainable lifestyles adopted by a significant portion of the younger generation.

There have been few studies on YOLO behaviors in the literature. To our best knowledge, our paper is the first one to investigate what preference factors determine YOLO lifestyles by comprehensively analyzing consumption and labor-leisure choice. Heimer et al. (2019) consider the impact of individuals' subjective mortality beliefs about life-cycle behavior on consumption with a CRRA utility setup. The key idea of their model is that the mortality belief enters the *effective* subjective discount rate. While they did not formally define YOLO, whether an individual is YOLO-like or not depends on the mortality belief in Heimer et al. (2019), which means that eventually the size of the subjective discount rate only determines the consumption pattern. Our paper is more comprehensive in that we consider the labor-leisure choice and characterize the lifestyles by both EIS and time preference.

Thanks to the rapid development of online platforms such as Uber, UberEats, and Lyft, more and more workers are having the choice of flexible workings through online platforms, particularly taking those flexible jobs as secondary jobs. For example, Hall and Krueger (2018) document that most of Uber's driver-partners had full- or part-time employment before joining Uber and reported that they are attracted by the flexibility. After starting to work with Uber, many of them continue to hold those previous job positions. Campbell (2018) reports that only one-third of ride-share drivers earn the most or all of their income from driving. In addition, workers move in and out of secondary jobs frequently, which is associated with large changes in working hours (Paxson and Sicherman (1996) and Renna and Oaxaca (2006)). The theoretical contribution of our paper is to provide a rigorous framework in continuous time to formalize this recent trend of the flexibility of labor-leisure choice. To do so, our framework combines two different lines of literature: canonical models of consumption and labor-leisure choice problem with the power utility setup (Bodie et al. (1992), Bodie et al. (2004), Farhi and Panageas (2007), and Choi et al. (2008)) and standard models of recursive utility without labor-leisure choice (e.g., Schroder and Skiadas (1999, 2003, 2005), Kraft et al. (2013, 2017), Matoussi and Xing (2018), and Melnyk et al. (2020)). There is a large literature in economics and finance considering optimal labor choice with recursive preference.[6] However, this literature mostly studies discrete time models without closed-form solutions. In addition, our focus relative to these papers is to investigate the

[6]For example, see Altug et al. (2020); Ai et al. (2018); Caldara et al. (2018); Gourio (2012) and references therein.

long-run implications of the agent's preference, in particular, the agent's EIS, which is a novel contribution to the literature.

On the technical contribution to the aforementioned literature, our model is minimal and stringent for considering a heterogeneous labor supply, but the labor-leisure choice setup makes the analysis mathematically quite challenging. To obtain analytic solutions, we take a simplified duality approach to derive the non-linear ODEs (Ordinary Differential Equations) with a free boundary that corresponds to the wealth level below which the agent provides more than the minimum labor supply. Then, the solution to the free-boundary non-linear ODEs can be expressed as the solutions to a system of integral equations in which the free boundary should be determined implicitly and the domains of the solution function vary depending on the free boundary. Moreover, the integral equations also include the function derivatives. Instead of handling this system of integral equations directly, we introduce a new transform that has several advantages. First, using our new transform, we fix the domain of the unknown function without the free boundary. Second, the system of integral equations obtained using the transform does not include the function derivatives. These features of our new transform help the verification and the solution analysis, as well as the development of a fast and stable numerical scheme.[7] Overall, our paper provides a novel technical contribution to solving free-boundary problems arising from the recursive utility setup.

The remainder of the paper proceeds as follows. Section 2 explains the model. Section 3 presents the solution analysis. The optimal policies and their long-run implications are investigated in Section 4. Section 5 investigates the wealth effect. The policy implications are discussed in Section 6. Section 7 provides the concluding remarks. All the proofs and the numerical scheme are in the Online Appendix.

[7]Note that in general the complexity and the inaccuracy of a numerical scheme dramatically increase as the step size for numerical computation decreases when discretizing free boundary-value problems because of the existence of the unknown free boundary and the derivative of the unknown function value at each node. In that sense, a numerical scheme without having the free boundary and the function derivatives is much more computationally stable.

2 Model

Financial Market: Consider an infinite horizon and continuous setup on a standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is generated by the standard Brownian motion B_t. In the financial market there are a risk-free asset and a risky asset. The risk-free rate r is a positive constant and the risky asset evolves $dS_t/S_t = \mu_S dt + \sigma_S dB_t$, where μ_S and σ_S are constant coefficients. The market price of risk is $\kappa \triangleq (\mu_S - r)/\sigma_S$, and it is assumed that $\kappa > 0$.

Preference: We consider a problem of an infinitely-lived agent who chooses consumption, risky investment, and leisure-labor over time. The agent's consumption and leisure rates are denoted by c_t and l_t, respectively. The agent has the Epstein-Zin recursive preference in which the utility process V_t associated with (c_t, l_t) is defined by

$$V_t = \mathbb{E}_t\left[\int_t^\infty f(s, c_s, l_s, V_s)ds\right], \tag{2.1}$$

where $\mathbb{E}_t[\cdot]$ is a conditional expectation over filtration $\mathcal{F}_t$ and the aggregator $f(s, c, l, V)$ is given by

$$f(s, c, l, V) = e^{-\rho s}\frac{(c^\eta l^{1-\eta})^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}}((1-\gamma)V)^{1-\theta}, \tag{2.2}$$

where $\theta \triangleq \dfrac{1-\psi^{-1}}{1-\gamma}$. The parameters $\rho > 0$, $\psi > 0$ $(\psi \neq 1)$, $\gamma > 0$ $(\gamma \neq 1)$, and $\eta \in (0, 1)$ represent the subjective discount rate, the coefficients of the elasticity of intertemporal substitution (EIS), the degree of relative risk aversion, and the elasticity of consumption, respectively. If $\psi = 1/\gamma$ (equivalently $\theta = 1$), the utility function collapses to the Cobb-Douglas utility function with consumption and leisure as in Farhi and Panageas (2007).

The aggregator in (2.2) is based on the discounted form aggregator proposed by Herdegen et al. (2023a). In the infinite horizon setup with the recursive preference, for a given consumption stream, the existence and uniqueness of the utility process satisfying (2.1) should be guaranteed before considering optimization. They are thoroughly discussed by Herdegen et al. (2023a) that using discounted form aggregator is more beneficial than the classical difference form aggregator. As in Herdegen et al. (2023a) and companion papers Herdegen et al. (2023b, 2021), we assume that $\theta > 0$ throughout the paper to guarantee the existence of our problem with the labor-leisure choice.[8] See Appendix A for more discussion on the aggregator and the assumption $\theta > 0$.

Labor-Leisure Choice: At each point of time t, the agent determines (l_t, h_t), a pair of the leisure and labor rates such that $l_t + h_t = \bar{L}$, where $\bar{L}$ represents the maximum available amount given at t. Note that we will normalize $\bar{L} = 1$ when we perform numerical and simulation analyses later. We use $\bar{L}$ for the general theoretical analysis. If the agent chooses h_t at $[t, t + dt)$, the agent receives an income stream $w_t h_t dt$, where w_t is a wage rate.

So far, the above labor-leisure choice setup is similar to that in Bodie et al. (1992). However, we add a more interesting feature of the labor supply trend in our model. Recently, more and more workers have gotten a secondary job to earn extra income and these jobs provide flexibility in working hours (see Paxson and Sicherman (1996), Renna and Oaxaca (2006), Hall and Krueger (2018) and Campbell (2018); see also our literature review for details). In this case, a worker must spend a certain *fixed* portion of hours in the primary and regular job and allocate the rest of hours between extra working and leisure. For example, if a teacher spends 8 hours a day at school and works as an UberEats driver for some hours after leaving school office, she earns the regular income as a teacher and the extra income as a driver.[9] It is notable that the working hours for the extra job are usually flexibly chosen (which is generally not the case of the primary job). As an example of the agent's labor-leisure supply problem, we can consider a case in which one member of a married couple has a regular job and the spouse flexibly chooses to provide extra labor depending on the agent's financial situation.

To model the above choice problem, we assume that there is a maximum leisure rate L, i.e., $0 < l_t \leq L$ ($L < \bar{L}$) and thus $\bar{L} - L$ represents the regular or base working hours from the primary job. By the same token, we suppose that there is a minimum working hours, i.e., $h_t \geq \bar{L} - L$. If $h_t = \bar{L} - L$, it implies that the worker does not choose extra work at t. In summary, the total labor income, $w_t h(t)$, is decomposed as the regular income and the extra

[8]Note that $\theta < 0$ when $\psi > 1$ and $\gamma > 1$. Thus, the case with $\psi > 1$ and $\gamma > 1$ is excluded to guarantee the existence of solution.

[9]In general, the wage rates for the primary job and the secondary job are different. For simplicity, we assume that the two rates are the same. The theoretical results, however, are robust to the extension case in which the two rates are different. Only some quantitative results are different depending on the difference between the two rates.

income as follows:

$$w_t h_t = w_t(\bar{L} - l_t) = \underbrace{w_t(\bar{L} - L)}_{\text{regular income}} + \underbrace{w_t(L - l_t)}_{\text{extra income}}.$$

From what follows, we assume that the wage rate is constant: $w_t = w$, for simplicity.

Agent's Problem: If we denote the investment amount in the risky asset at time t by π_t, the agent's wealth dynamics, X_t, at time $t > 0$ is governed by

$$dX_t = (rX_t + (\mu_S - r)\pi_t - c_t + w(\bar{L} - l_t))dt + \sigma_S \pi_t dB_t, \qquad X_0 = x. \tag{2.3}$$

Let us define that (c, l, π) belongs to the admissible class $\mathcal{A}(x)$ if it is progressively measurable with respect to $\mathcal{F}_t$ and satisfies the following conditions: (i) $\int_0^t c_s ds < \infty$ a.s., $\int_0^t l_s ds < \infty$ a.s., and $\int_0^t \pi_s^2 ds < \infty$ a.s. for all $t > 0$, and (ii) The wealth process X_t in (2.3) that corresponds to (c, l, π) satisfies $X_t > -\frac{w\bar{L}}{r}$.

Then, the agent's problem is to optimally select the consumption rate c, investment amount π, and the leisure rate l in order to maximize the recursive utility in (2.1) with aggregator in (2.2). Thus, the agent's problem is

$$\max_{(c_t, l_t, \pi_t) \in \mathcal{A}(x)} \mathbb{E}\left[\int_0^\infty f(t, c_t, l_t, V_t) dt\right]. \tag{2.4}$$

3 Analysis

Note that there are two major challenges when deriving the solution. The first one is the recursive utility structure. The second and more important challenge comes from the flexibility of the labor-leisure choice. As will be shown in what follows, the labor-leisure choice setup makes the analysis quite complicated, but provides interesting and rich implications.

3.1 Non-linear ODEs and Free Boundary

Before we proceed, we define $\gamma_1 \triangleq 1 - \eta(1 - \psi^{-1})$ and $\gamma_2 \triangleq 1 - \eta(1 - \gamma)$. Note that γ_1 coincides with γ_2 when $\theta = 1$. By applying the dynamic programming principle, the HJB

(Hamilton-Jacobi-Bellman) equation associated with the value function is[10]

$$\frac{\rho}{\theta}V = \max_{\{c,l,\pi\}} \Big[\frac{(c^\eta l^{1-\eta})^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}}((1-\gamma)V)^{1-\theta} + (rx + w(\bar{L}-l) + \kappa\sigma_S\pi - c)V_x + \frac{1}{2}\pi^2\sigma_S^2 V_{xx}\Big]. \tag{3.1}$$

Intuitively, the optimal leisure is increasing in the wealth level. Thus, the agent enjoys the full leisure level L and has minimum working hours when the wealth level is sufficiently high so that it is greater than or equal to $\bar{x}$, the *threshold of wealth* for full leisure. If the wealth level is less than $\bar{x}$, the optimal leisure level is less than L.

First consider the case in which $x < \bar{x}$ and thus the optimal leisure is less than L. The first-order conditions imply that the candidates for the optimal consumption rate, leisure rate, and investment amount are given by

$$l^*(x) = \eta^\psi \left(\frac{\eta w}{1-\eta}\right)^{-\gamma_1\psi} \Big\{((1-\gamma)V)^{\theta-1}V_x\Big\}^{-\psi}, \tag{3.2}$$

$$c^*(x) = \frac{\eta w}{1-\eta} l^*(x), \quad \pi^*(x) = -\frac{\kappa V_x}{\sigma_S V_{xx}}, \tag{3.3}$$

respectively, for given wealth level $x < \bar{x}$. By substituting these into the HJB equation (3.1), we have the following non-linear ODE for the value function V for $x < \bar{x}$:

$$0 = \frac{\eta^\psi}{\eta(\psi-1)}\left(\frac{\eta w}{1-\eta}\right)^{1-\gamma_1\psi} ((1-\gamma)V)^{-(\theta-1)\psi} V_x^{1-\psi} - \frac{\rho}{\theta}V - \frac{1}{2}\kappa^2\frac{V_x^2}{V_{xx}} + (rx + w\bar{L})V_x. \tag{3.4}$$

Now let us consider the case in which $x \geq \bar{x}$. In this case, the optimal leisure is $l^*(x) = L$, and the candidate for the optimal consumption rate is

$$c^*(x) = \left[\frac{L^{\psi^{-1}-\gamma_1}}{\eta}((1-\gamma)V)^{\theta-1}V_x\right]^{-\frac{1}{\gamma_1}}. \tag{3.5}$$

[10] More precisely, since the aggregator has the time term, we first define the value function of problem (2.4) by $v(t,x)$. Then, $v(t,x)$ satisfies the following HJB equation:

$$0 = \max_{\{c,l,\pi\}} f(t,c,l,v) + v_t + (rx + w(\bar{L}-l) + \pi(\mu_S - r) - c)v_x + \frac{1}{2}\pi^2\sigma_S^2 v_{xx}.$$

By setting $v(t,x) = e^{-\frac{\rho}{\theta}t}V(x)$, we have (3.1).

The candidate form of the optimal investment amount is given the same as in (3.3). By substituting the candidate policies into the HJB equation (3.1), we obtain the following nonlinear ODE for the value function V for $x > \bar{x}$:

$$0 = \frac{\gamma_1}{1-\gamma_1}\eta^{\frac{1}{\gamma_1}} L^{1-\frac{1}{\gamma_1 \psi}} \left((1-\gamma)V\right)^{(\theta-1)(-\frac{1}{\gamma_1})} V_x^{1-\frac{1}{\gamma_1}} - \frac{\rho}{\theta}V - \frac{1}{2}\kappa^2 \frac{V_x^2}{V_{xx}} + (rx + w(\bar{L} - L))V_x. \tag{3.6}$$

Note that the value function V should be C^1, i.e., continuously differentiable at $x = \bar{x}$. This fact, along with the above ODEs for V, will be used to determine the full leisure threshold of wealth $\bar{x}$.

Now we transform the value function V to a new function $\varphi(z)$ with a new variable z that satisfies the following relationship with V and x:

$$V(x) = \varphi(z) - z\varphi'(z), \quad x = -\varphi'(z). \tag{3.7}$$

Remark 1 (Simplified Duality Approach)**.** *The transform in* (3.7) *is inspired by the standard duality method. Let $J(z)$ be the dual value function obtained from the standard dual problem with additive-separable utility. Then, $J(z)$ and the primal value function $V(x)$ has the duality relationship: $V(x) = \inf_{z>0}(J(z) + zx)$. From the first-order condition, we have the relationship between the primal variable x and the dual variable z as $x = -J'(z)$, and it follows that $V(x) = J(z) - zJ'(z)$.*

Note that the approach summarized in Remark 1 in general works well not only for the additive-separable utility case but also for the recursive utility setup. From definition (3.7), we obtain

$$V_x = z \quad \text{and} \quad V_{xx} = -\frac{1}{\varphi''(z)}. \tag{3.8}$$

Given the concavity of $V(x)$ (equivalently the convexity of $\varphi(z)$), there is the inverse relationship between x and the new variable z defined implicitly by (3.7): $x < \bar{x}$ ($x \geq \bar{x}$) corresponds to $z > \bar{z}$ ($0 < z \leq \bar{z}$), where $\bar{z}$ is defined implicitly by $\bar{x} = -\varphi'(\bar{z})$.

Let us define $\varphi_1(z) \triangleq \varphi(z)$ for $z \geq \bar{z}$ and $\varphi_2(z) \triangleq \varphi(z)$ for $0 < z \leq \bar{z}$. Then, we can derive the non-linear ODEs for $\varphi_1(z)$ and $\varphi_2(z)$ as follows:

$$0 = \frac{\eta^{\psi}}{\eta(\psi-1)} \left(\frac{\eta w}{1-\eta}\right)^{1-\gamma_1\psi} \left((1-\gamma)(\varphi_1(z) - z\varphi_1'(z))\right)^{-(\theta-1)\psi} z^{1-\psi} \tag{3.9}$$
$$- \frac{\rho}{\theta}(\varphi_1(z) - z\varphi_1'(z)) + \frac{1}{2}\kappa^2 z^2 \varphi_1''(z) + w\bar{L}z - rz\varphi_1'(z), \quad z > \bar{z},$$
$$0 = \frac{\gamma_1}{1-\gamma_1}\eta^{\frac{1}{\gamma_1}} L^{1-\frac{1}{\gamma_1\psi}} \left((1-\gamma)(\varphi_2(z) - z\varphi_2'(z))\right)^{-\frac{\theta-1}{\gamma_1}} z^{1-\frac{1}{\gamma_1}} \tag{3.10}$$
$$- \frac{\rho}{\theta}(\varphi_2(z) - z\varphi_2'(z)) + \frac{1}{2}\kappa^2 z^2 \varphi_2''(z) + w(\bar{L} - L)z - rz\varphi_2'(z), \quad 0 < z < \bar{z}.$$

Recall that the value function V should be C^1 at $x = \bar{x}$, which leads to the following free-boundary conditions at $z = \bar{z}$:

$$\varphi_1(\bar{z}) = \varphi_2(\bar{z}), \quad \varphi_1'(\bar{z}) = \varphi_2'(\bar{z}). \tag{3.11}$$

Once we obtain the solutions $\varphi_1(z)$ and $\varphi_2(z)$ to the ODEs (3.9) and (3.10), respectively, and the free boundary $\bar{z}$ satisfying (3.11), the full leisure threshold of wealth $\bar{x}$ will be determined as

$$\bar{x} = -\varphi_1'(\bar{z}) = -\varphi_2'(\bar{z}). \tag{3.12}$$

Remark 2. *In the case of additive-separable utility, the benefit of simplified duality approach in* (3.7) *is that, in contrast to the highly non-linear ODE for the primal value function $V(x)$, the ODE for $\varphi(z)$ is a linear ODE. In our model with recursive preference, although the ODEs for $\varphi_1(z)$ and $\varphi_2(z)$ in* (3.9) *and* (3.10) *are not linear, they only have one additional non-linear term, which makes the analysis more tractable.*

3.2 Solution

Before deriving the solutions to ODEs (3.9) and (3.10), let $n_+ > 1$ and $n_- < 0$ be the two real roots of the following quadratic equation:[11]

$$q(n) \triangleq \frac{1}{2}\kappa^2 n^2 + \left(\frac{\rho}{\theta} - r - \frac{1}{2}\kappa^2\right) n - \frac{\rho}{\theta} = 0.$$

[11] Since $q(0) = -\frac{\rho}{\theta} < 0$ and $q(1) = -r < 0$, it is guaranteed that $n_+ > 1$ and $n_- < 0$.

We also define K and K_2 as follows and assume that both of them are positive:[12] $K \triangleq -q\left(1-\frac{1}{\gamma}\right) > 0, \quad K_2 \triangleq -q\left(1-\frac{1}{\gamma_2}\right) > 0$. Here, we can verify that particular solutions to ODEs (3.9) and (3.10) are

$$\varphi_{p,1}(z) = \frac{\gamma}{1-\gamma} A_1 z^{1-\frac{1}{\gamma}} + \frac{w\bar{L}}{r} z, \quad \varphi_{p,2}(z) = \frac{\gamma_2}{1-\gamma_2} A_2 z^{1-\frac{1}{\gamma_2}} + \frac{w(\bar{L}-L)}{r} z,$$

respectively, where

$$A_1 \triangleq \left[\frac{\eta^{\psi}}{\psi\gamma K\eta\theta}\left(\frac{\eta w}{1-\eta}\right)^{1-\gamma_1\psi}\right]^{\frac{1}{1+(\theta-1)\psi}}, \quad A_2 \triangleq \left[\frac{1-\gamma_2}{\gamma_2}\frac{\gamma_1}{1-\gamma_1}\frac{\eta^{\frac{1}{\gamma_1}} L^{1-\frac{1}{\gamma_1\psi}}}{K_2}\eta^{\frac{\theta-1}{\gamma_1}}\right]^{\frac{\gamma_1}{\gamma_1+(\theta-1)}}. \tag{3.13}$$

Then, the general solutions to ODEs (3.9) and (3.10) are given as

$$\varphi_1(z) = \alpha_1(z) z^{n_+} + \beta_1(z) z^{n_-} + \varphi_{p,1}(z), \quad z \geq \bar{z}, \tag{3.14}$$

$$\varphi_2(z) = \alpha_2(z) z^{n_+} + \beta_2(z) z^{n_-} + \varphi_{p,2}(z), \quad 0 < z \leq \bar{z}, \tag{3.15}$$

subject to

$$\alpha_i'(z) z^{n_+} + \beta_i'(z) z^{n_-} = 0 \quad \text{for } i = 1, 2. \tag{3.16}$$

Note that the explicit solution forms for $\alpha_i(z)$ and $\beta_i(z)$ are provided in Proposition A.1 located in Appendix A with the concrete analysis for its derivation process. Then, by using the relationship between the value function and the solutions to the ODEs (3.9) and (3.10), we reconstruct the value function $V(x)$ in (2.4) as

$$V(x) = \begin{cases} \varphi_1(\mathcal{Z}_1(x)) - \mathcal{Z}_1(x)\varphi_1'(\mathcal{Z}_1(x)), & x \leq \bar{x} \\ \varphi_2(\mathcal{Z}_2(x)) - \mathcal{Z}_2(x)\varphi_2'(\mathcal{Z}_2(x)), & x \geq \bar{x}, \end{cases}$$

where $\mathcal{Z}_i(x)$ is implicitly defined by $x = -\varphi_i'(\mathcal{Z}_i(x))$ for $i = 1, 2$, and $\bar{x} = -\varphi_1'(\bar{z}) = -\varphi_2'(\bar{z})$.

Now we are ready to state the analytic form of the agent's optimal policy.

[12] $K > 0$ and $K_2 > 0$ imply that $n_- < 1 - \frac{1}{\gamma} < n_+$ and $n_- < 1 - \frac{1}{\gamma_2} < n_+$, respectively.

Proposition 1. *The optimal investment $\pi^*(x)$, consumption $c^*(x)$, and leisure $l^*(x)$ are given by*

$$\pi^*(x) = \begin{cases} \dfrac{\kappa}{\sigma_S}\mathcal{Z}_1(x)\varphi_1''(\mathcal{Z}_1(x)), & x \le \bar{x}, \\ \dfrac{\kappa}{\sigma_S}\mathcal{Z}_2(x)\varphi_2''(\mathcal{Z}_2(x)), & x \ge \bar{x}, \end{cases} \tag{3.17}$$

$$c^*(x) = \begin{cases} \frac{\eta w}{1-\eta} l^*(x), & x \le \bar{x}, \\ \frac{\eta w}{1-\eta} L \left[\frac{F_2\left(\frac{\mathcal{Z}_2(x)}{\bar{z}}\right)}{F_2(1)}\right]^{-\frac{\theta-1}{\gamma_1}} \left(\frac{\mathcal{Z}_2(x)}{\bar{z}}\right)^{-\frac{1}{\gamma_2}}, & x \ge \bar{x}, \end{cases} \tag{3.18}$$

$$l^*(x) = \begin{cases} L \left[\frac{F_1\left(\frac{\mathcal{Z}_1(x)}{\bar{z}}\right)}{F_1(1)}\right]^{-(\theta-1)\psi} \left(\frac{\mathcal{Z}_1(x)}{\bar{z}}\right)^{-\frac{1}{\gamma}}, & x \le \bar{x}, \\ L, & x \ge \bar{x}, \end{cases} \tag{3.19}$$

where the functional forms of F_1 and F_2 are given in Appendix A.

We will investigate the implications of the optimal policies and characterize the types of agents by using the optimal behaviors in Section 4. Before it, for the sake of comparison, we first consider two special cases in Section 3.3.

3.3 Special Cases

Here we consider the following two special cases: (i) the additive-separable utility case (when $\psi = 1/\gamma$) and (ii) the recursive utility with fixed working hours ($l_t = l_0$ for constant $l_0 > 0$).

Special Case I (additive-separable utility): Recall that when $\psi = 1/\gamma$, the recursive utility is reduced to the Cobb-Douglas utility function of consumption and leisure. Consequently, the ODEs (3.9) and (3.10) become linear ODEs. Then, we can derive the explicit solutions similar to those in Choi et al. (2008). Note that $\gamma_1 = \gamma_2$ and $\theta = 1$ when $\psi = 1/\gamma$. Thus, we add the superscript $\theta = 1$ to the notation when $\psi = 1/\gamma$.

We can verify that the free boundary becomes[13] $\bar{z}^{\theta=1} = \zeta^{\theta=1} = \rho\left(\frac{\eta w}{1-\eta}\right)^{-\gamma_2} L^{-\gamma}$. In addition, α_i's and β_i's become constants and especially $\alpha_1^{\theta=1} = \beta_2^{\theta=1} = 0$. The solutions to

[13]More precisely, we use (A.5) and (A.6) to derive the result.

ODEs (3.9) and (3.10) when $\theta = 1$ are given as

$$\varphi_1^{\theta=1}(z) = \beta_1^{\theta=1} z^{n_-} + \frac{\gamma}{1-\gamma} A_1^{\theta=1} z^{1-\frac{1}{\gamma}} + \frac{w\bar{L}}{r} z, \quad z \geq \bar{z}^{\theta=1},$$
$$\varphi_2^{\theta=1}(z) = \alpha_2^{\theta=1} z^{n_+} + \frac{\gamma_2}{1-\gamma_2} A_2^{\theta=1} z^{1-\frac{1}{\gamma_2}} + \frac{w(\bar{L}-L)}{r} z, \quad 0 < z \leq \bar{z}^{\theta=1},$$

where

$$\beta_1^{\theta=1} = -\frac{(n_+ - 1 + \frac{1}{\gamma})(\frac{1}{\gamma} - \frac{1}{\gamma_2})}{(n_+ - n_-)(1 - n_- - \frac{1}{\gamma_2})(1-n_-)(1-\gamma)} A_1^{\theta=1} \left(\bar{z}^{\theta=1}\right)^{1-n_- - \frac{1}{\gamma}} < 0, \quad (3.20)$$
$$\alpha_2^{\theta=1} = \frac{(1 - n_- - \frac{1}{\gamma_2})(\frac{1}{\gamma} - \frac{1}{\gamma_2})}{(n_+ - n_-)(n_+ - 1 + \frac{1}{\gamma})(n_+ - 1)(1-\gamma_2)} A_2^{\theta=1} \left(\bar{z}^{\theta=1}\right)^{-(n_+ - 1 + \frac{1}{\gamma_2})} > 0, \quad (3.21)$$

with $A_1^{\theta=1} = \frac{\eta^{\frac{1}{\gamma}}}{K^{\theta=1}\eta} \left(\frac{\eta w}{1-\eta}\right)^{1-\frac{\gamma_2}{\gamma}}$ and $A_2^{\theta=1} = \frac{\eta^{\frac{1}{\gamma_2}} L^{1-\frac{\gamma}{\gamma_2}}}{K_2^{\theta=1}}$.

Corollary 1 (When $\psi = 1/\gamma$). *By setting $\psi = 1/\gamma$, the optimal consumption, leisure, and investments when $\theta = 1$ are given as*

$$c^{\theta=1}(x) = \begin{cases} K^{\theta=1}\eta \left(x + \frac{w\bar{L}}{r} + n_- \beta_1^{\theta=1} \mathcal{Z}_1^{\theta=1}(x)^{n_- - 1}\right), & x \leq \bar{x}^{\theta=1}, \\ K_2^{\theta=1} \left(x + \frac{w(\bar{L}-L)}{r} + n_+ \alpha_2^{\theta=1} \mathcal{Z}_2^{\theta=1}(x)^{n_+ - 1}\right), & x \geq \bar{x}^{\theta=1}, \end{cases}$$
$$l^{\theta=1}(x) = \begin{cases} \frac{K^{\theta=1}(1-\eta)}{w} \left(x + \frac{w\bar{L}}{r} + n_- \beta_1^{\theta=1} \mathcal{Z}_1^{\theta=1}(x)^{n_- - 1}\right), & x \leq \bar{x}^{\theta=1}, \\ L, & x \geq \bar{x}^{\theta=1}, \end{cases}$$
$$\pi^{\theta=1}(x) = \begin{cases} \frac{\kappa}{\sigma_S} \mathcal{Z}_1^{\theta=1}(x) (\varphi_1^{\theta=1})''(\mathcal{Z}_1^{\theta=1}(x)), & x \leq \bar{x}^{\theta=1}, \\ \frac{\kappa}{\sigma_S} \mathcal{Z}_2^{\theta=1}(x) (\varphi_2^{\theta=1})''(\mathcal{Z}_2^{\theta=1}(x)), & x \geq \bar{x}^{\theta=1}, \end{cases}$$

where $\mathcal{Z}_i^{\theta=1}(x)$ is defined implicitly by $x = -(\varphi_i^{\theta=1})'(\mathcal{Z}_i^{\theta=1}(x))$ for $i = 1, 2$, and $\bar{x}^{\theta=1} = -(\varphi_1^{\theta=1})'(\bar{z}^{\theta=1}) = -(\varphi_2^{\theta=1})'(\bar{z}^{\theta=1})$.

There is an important remark on the optimal policies for a special case when $\theta = 1$. Both optimal consumption and leisure increase with ρ (see Figure 1). In other words, the agent tends to consume more and enjoy more leisure as the level of the agent's subject discounting increases. This result is generally true for the recursive utility case. However, the quantitative impact of ρ will significantly differ by ψ, as will be seen later.

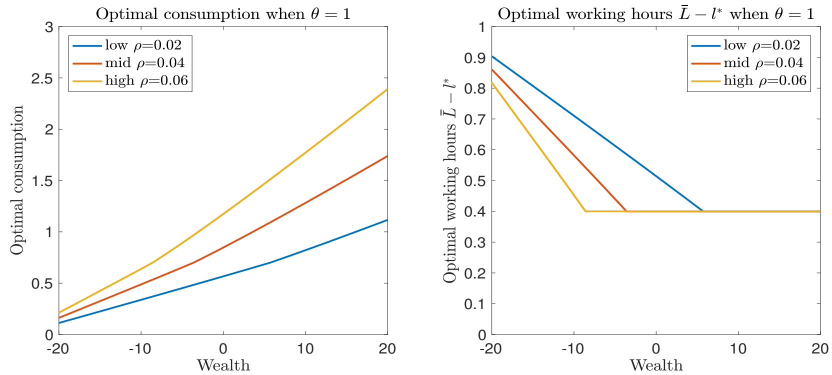

Figure 1: The optimal consumption and working hours for the additive-separable utility case ($\theta = 1$). The other common parameters are $r = 0.02,\ \kappa = 0.25,\ \eta = 0.7,\ \bar{L} = 1,\ L = 0.6,\ w = 0.5,\ \gamma = 2$.

Special Case II (recursive utility with fixed working hours): Suppose the agent cannot adjust their working hours. In this case, the solution turns out to be the same as the case in which the risk aversion parameter is given by γ_2. Moreover, the following corollary shows how risk investment and consumption are affected by EIS.

Corollary 2. *Suppose the agent cannot adjust their working hours, i.e., $l_t = l_0$ for some constant l_0 for all $t \geq 0$. The optimal investment and consumption are given as*

$$\pi_0(x) = \frac{\kappa}{\sigma_S \gamma_2}\left(x + \frac{w(\bar{L} - l_0)}{r}\right), \quad c_0(x) = K_0\left(x + \frac{w(\bar{L} - l_0)}{r}\right),$$

where $K_0 = \left(\rho - r - \frac{\kappa^2}{2\gamma_2}\right)\frac{1}{\gamma_1} + r + \frac{\kappa^2}{2\gamma_2}$. Then, the following results hold:

(a) The risky investment is independent of ψ.

(b) If $\dfrac{\kappa^2}{2\gamma_2} + r > \rho$, the substitution effect dominates, i.e., consumption decreases as ψ increases.

(c) If $\dfrac{\kappa^2}{2\gamma_2} + r < \rho$, the income effect dominates, i.e., consumption increases as ψ increases.

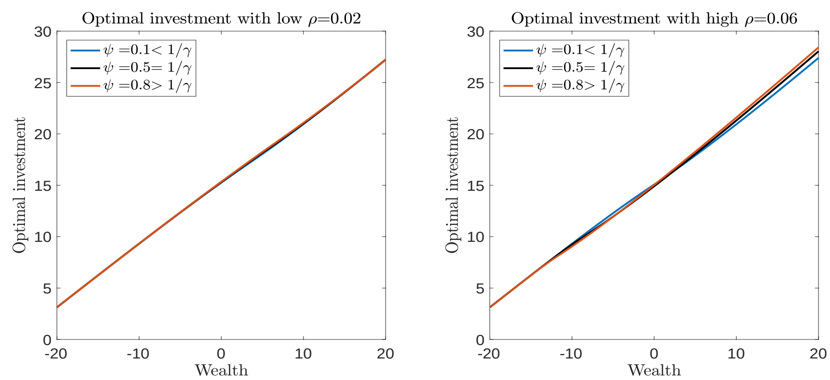

Figure 2: Optimal risky investment. The impact of EIS on risky investments is fairly small. The other parameters are $r = 0.02,\ \kappa = 0.25,\ \eta = 0.7,\ w = 0.5,\ \gamma = 2,\ \bar{L} = 1,\ L = 0.6$. The parameters for the market and the preference are chosen from widely used ones in the various literature. The total available time for leisure and labor is normalized as $\bar{L} = 1$. The minimum working hours is set as $\bar{L} - L = 40\%$ of the total available time. This parameter configuration is used for all the figures and examples in the paper.

4 Optimal Policies and Implications in the Long Run

4.1 Discussion of Optimal Policies

Now let us investigate the implications for the optimal investment (3.17), consumption (3.18), and leisure (3.19) given in Proposition 1 in detail. In particular, we do so by comparing them with those from the case in which the labor supply is fixed.

First, regarding the optimal portfolio (3.17), Corollary 2(a) is a well-known result when the labor supply is fixed (Bhamra and Uppal, 2006; Svensson, 1989; Cai et al., 2018): If the investment opportunity is constant, i.e., μ, σ, and r are constant, then the EIS does not affect the risky investment when the labor supply is not flexible. On the contrary, the risky investment depends on ψ in the case in which the agent optimally adjusts working hours with the minimum working hour requirement. This is the immediate consequence from (3.17) given in Proposition 1.

Corollary 3. π_t^* *of* (3.17) *is not constant with respect to* ψ.

The intuition for Corollary 3 is as follows. If financial wealth is sufficiently high (i.e., $X_t > \bar{x}$), the agent works for the minimum amount of hours. If $X_t < \bar{x}$, the agent works more. This change according to the wealth level implies that the *effective* EIS defined by the value

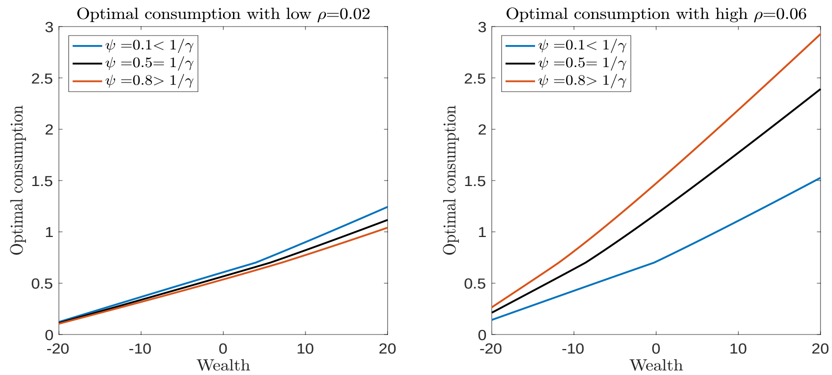

Figure 3: Optimal consumption as a function of wealth. The effects of EIS on consumption are opposite according to the degree of the patience (ρ). The left panel shows that consumption decreases in ψ when the agent is patient ($\rho = 0.02$). The right panel shows that consumption increases in ψ when the agent is impatient ($\rho = 0.06$). The other parameters are the same as in Figure 2.

function (not by the utility function), i.e., $\frac{-xV''(x)}{V'(x)}$, changes as x does, which makes the optimal risky portfolio depend on EIS. Observe from (3.3) that the effective EIS is inversely related to the optimal portfolio.

However, it is important to note that the quantitative impact is limited, as seen in Figure 2. This result implies that the existence of the free boundary $\bar{x} > 0$ has little effect on the risky investment. From the result of Cai et al. (2018), we see that the quantitative effect will still be very small even if we extend the current model to a more general case with a regime switching investment opportunity set or with the presence of taxes and transaction costs.

Second, in order to understand how EIS affects consumption (3.18), it is important to see whether the substitution effect dominates the income effect or vice versa. In the case of the non-flexible labor-leisure choice, Corollaries 2(b) and (c) show the exact conditions of which one dominates the other when the labor supply is fixed. If $\kappa^2/\gamma_2 + r > \rho$, it implies that the financial market summarized by the Sharpe ratio (κ) and the risk-free rate (r) is attractive relative to the rate of time preference or subject discounting (ρ). Therefore, as ψ increases, the agent consumes less and saves more in order to increase consumption in the future. The opposite case occurs when the agent is fairly impatient, i.e., when $\kappa^2/\gamma_2 + r < \rho$.

We find that a similar result still holds when the labor supply is flexibly adjusted, while the exact conditions are hard to obtain due to the complexity of the solution form. More

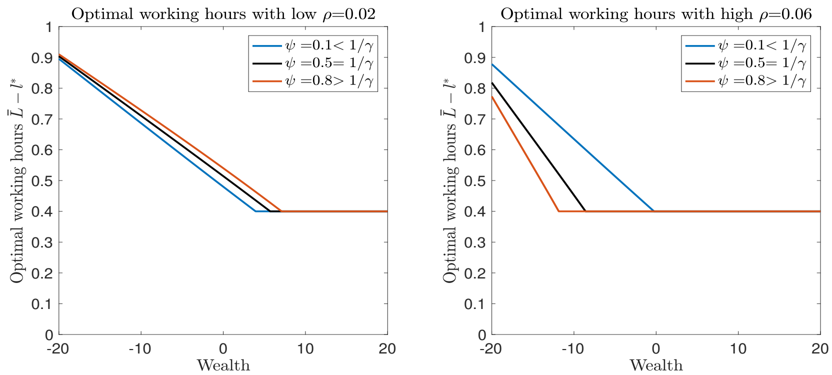

Figure 4: Optimal working hours as a function of wealth. In the left panel, optimal working hours are increasing in ψ when the agent is patient ($\rho = 0.02$). In the right panel, the optimal working hours are decreasing in ψ when the agent is impatient ($\rho = 0.06$). The other common parameters are the same as in Figure 2.

precisely, if ρ is relatively small, i.e., the financial market is attractive, then the substitution effect becomes stronger than the income effect as EIS increases: current consumption decreases with ψ (the left panel of Figure 3). In contrast, if ρ is sufficiently high, i.e., the financial market is not attractive relative to the agent's impatience, the income effect becomes stronger than the substitution effect as EIS increases: current consumption increases with ψ (the right panel of Figure 2).

Third and most important, regarding the optimal leisure (3.19), Figure 4 presents the optimal working hours. There are two things to note. First, the agent works more than the minimum working hours if and only if $x < \bar{x}$, where $\bar{x}$ is determined by (3.12). Second, $\bar{x}$ increases with ψ when the agent is patient because the substitution effect becomes stronger than the income effect as ψ increases (the left panel of Figure 4). This implies that if the agent is patient, he/she tends to work more as ψ increases. In contrast, $\bar{x}$ decreases with ψ when the agent is sufficiently impatient (the right panel of Figure 4). The reason is that in this case, as ψ increases, the income effect becomes stronger than the substitution effect. This implies that if the agent is impatient, he/she tends to work less as ψ increases. Figure 5 also confirms this property of $\bar{x}$ as a function of ψ.

As a summary of optimal policies, we can categorize agents into the following four types: (i) agents with (low ρ, low ψ), (ii) agents with (low ρ, high ψ), (iii) agents with (high ρ, low ψ), and (iv) agents with (high ρ, high ψ), as presented in Table 1. (iii) and (iv) are called

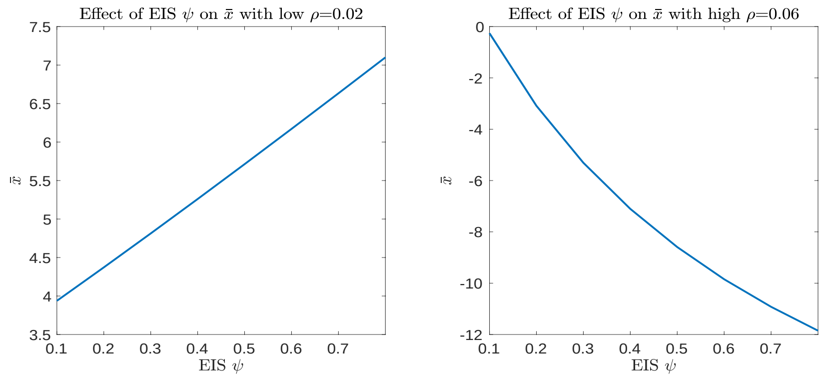

Figure 5: Threshold $\bar{x}$ according to ψ. In the left panel, $\bar{x}$ is increasing in ψ when the agent is patient ($\rho = 0.02$). In the right panel, $\bar{x}$ is decreasing in ψ when the agent is impatient ($\rho = 0.06$). The other parameters are the same as those in Figure 2.

YOLO types, and (i) and (ii) are called Non-YOLO types. Recall that the impact of ρ in the case of high ρ is opposite to that in the case of low ρ. Until now, it may not sound clear why we call (ii) and (iv) 'extreme' in addition to YOLO and Non-YOLO. We will explain in more detail in the following section by using the long-run implications of the optimal policies.

Before we close this subsection, note that among the three preference factors of the Epstein-Zin utility setup, i.e., (a) risk aversion, (b) EIS (Elasticity of Intertemporal Substitution), and (c) subjective discounting, we restrict our attention to (b) and (c) as determinants of whether an individual exhibits a YOLO-type behavioral pattern or not. This selection is motivated by two primary reasons. Firstly, while YOLO does involve an element of risk-taking, it is commonly associated with an *excessive* willingness to take risks, a characterization that does not align neatly with the rational and risk-averse foundation of an economic agent's model. Secondly, it is intuitive to see that (b) and (c) hold more relevance in guiding decisions concerning the allocation of resources between present and future consumption, and this allocation is mediated through the labor-leisure choice and, consequently, income generation.

4.2 Implications in the Long Run

So far we have investigated the optimal policies as a function of x. Using these baseline results in Section 4, we will further explore the dynamics of optimal policies over time.

More precisely, we will discuss the implications for wealth accumulation and the long-run sustainability of the consumption and labor-leisure choice for each type.

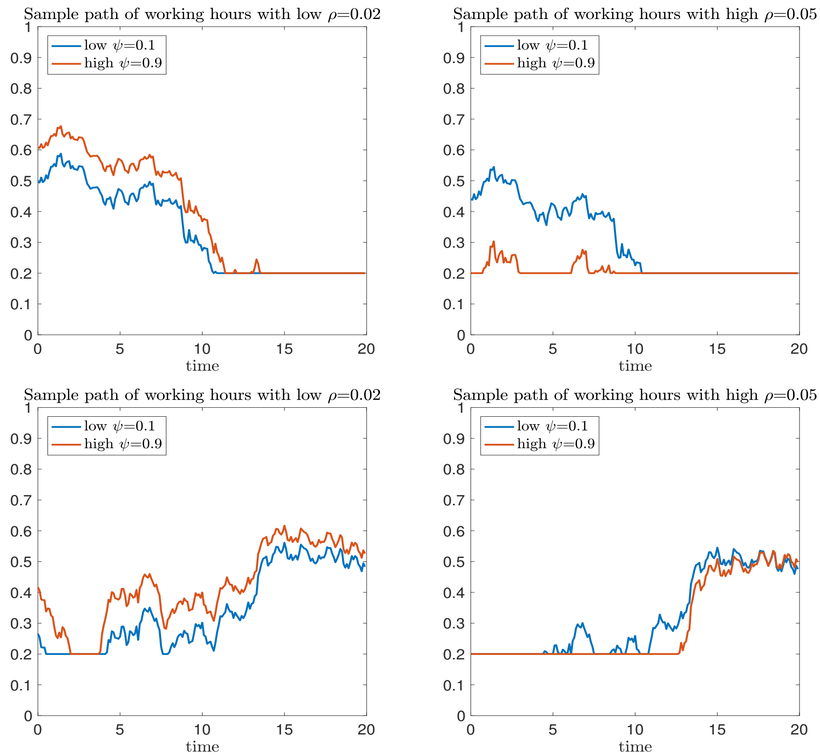

Figure 6: The sample paths of working hours: the left panels are when the substitution effect dominates (small ρ) and the right panels are when the income effect dominates (high ρ). The top panels are when the agent becomes wealthier over time and the bottom panels are when the agent becomes poorer over time. The other parameters are the same as those in Figure 2.

First, we present how the agent changes the labor supply over time. Figure 6 shows several sample paths of working hours. The upper panels are the cases where the agents start with a low level of wealth, but the underlying process (or the asset price in the financial market) increases on average in time: There are more and larger good shocks than bad shocks to agents' wealth in this sample path. Therefore, the agents are initially poor, but become richer over time so that the working hours eventually decrease to the minimum level. In contrast, the lower panels are the cases where the agents start with a high level of wealth, but the underlying process decreases on average in time in this sample path. Thus, the agents are initially rich, but become poorer and poorer over time so that the working hours generally increase in time. Note that for patient agents $\bar{x}$ increases with ψ, which means that extra working hours of a Type N_e agent are always higher than those of a Type N agent over time except when both agents are sufficiently rich: the red lines in general are higher than the blue one in the left panels in Figure 6. In contrast, for impatient agents $\bar{x}$ decreases with ψ, which means that extra working hours of a Type Y_e agent are mostly lower than those of a Type Y agent except when both agents are sufficiently rich: the red lines in general are lower than the blue one in the right panels of Figure 6.

While Figure 6 only shows several special cases, we find that these patterns are typical patterns over time and thus have significant implications for wealth accumulation and lifetime labor supply. To see the long-run effect of labor supply on wealth accumulation, we generate 10,000 sample paths of the underlying process for 35 years and obtain the average values of (i) the ratio of the extra working hours to the minimum working hours, $\frac{L-l_t^*}{\bar{L}-L}$, (ii) consumption/mHW, (iii) investment/mHW, and (iv) financial wealth over time. Here, we define the mHW (minimum Human Wealth) by the sum of current wealth (X) and the minimum human capital (mHC) that represents the present value of the minimum lifetime income: $mHW = X + mHC$, where

$$mHC = \int_0^\infty e^{-rt} w(\bar{L} - L)dt = \frac{(\bar{L} - L)w}{r}.$$

Figure 7 plots those four average values in time of two patient Non-YOLO agents with different EIS's (Type N and Type N_e) starting with the same initial wealth. In their early life, the Type N_e agent works harder, earns more income (the top-right panel) and consumes less (the bottom-left panel). Consequently, the Type N_e agent accumulates more wealth than the Type N agent does over time (the top-left panel). Note that their risky investment patterns are not much different (the bottom-right panel) over time. It is notable that the total working hours of the Type N_e agent gradually decrease in time while those of the Type N agent are quite flat (the top-right panel). Therefore, we interpret that the consumption and leisure pattern of Type N_e agents is more extreme. Note that the Type N_e agent accumulates more wealth over the life cycle thanks to less consumption and less leisure early in life.

In contrast, Figure 8 plots the same four average values in a time of YOLO agents (Type Y and Type Y_e) starting with the same initial wealth. In this case, as in the previous Non-YOLO case, one may suspect that the Type Y agent mostly works longer over the entire

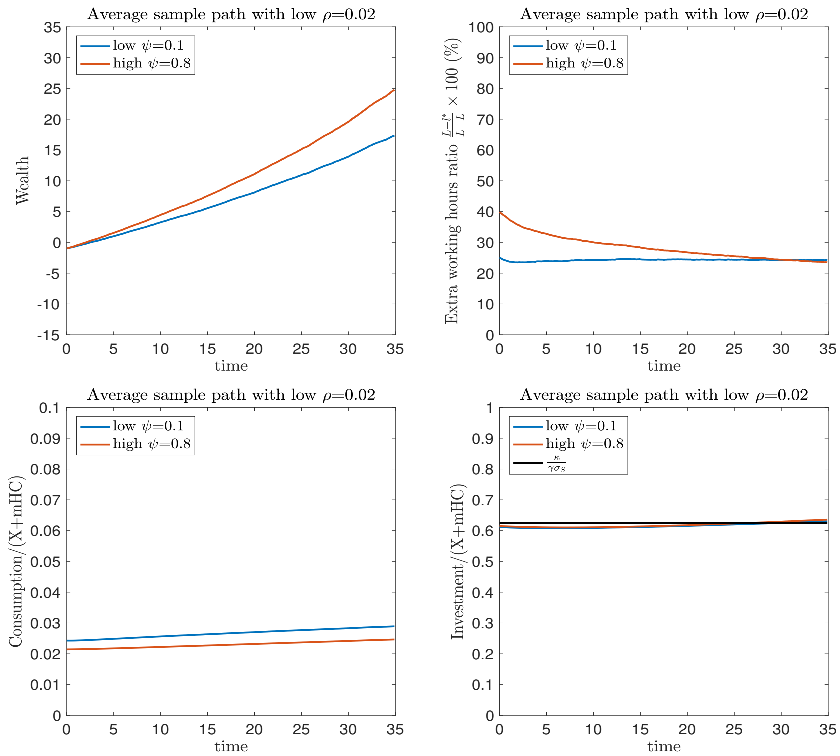

Figure 7: Average sample path over 35 years when the agent is patient ($\rho = 0.02$). The other parameters are the same as those in Figure 2.

lifetime than the Type Y_e agent does. However, strikingly, it is not the case. As seen in the top-right panel of Figure 8, in the early stage of life, the Type Y_e agents enjoy more leisure (the top-right panel) and consume a lot more in the proportion of wealth than the Type Y agent does (the bottom-left panel). As a consequence, the wealth level of the Type Y_e agent decreases on average over time while that of the Type Y agents gradually increases (the top-left panel). Then, the former's wealth level becomes significantly low at a certain point in life so he or she has to work longer and longer in order to maintain a higher consumption rate proportional to wealth. Again the risky investments between the two agents are not much different (the bottom-right panel). This fact confirms that the main reason why the Type Y_e agent has to work longer in later life is that he/she consumes too much and works less early in life so that his/her wealth level becomes smaller and smaller later in life, not from the return on investment being small.

So far, we have separately compared Types N and N_e and Types Y and Y_e with respect to

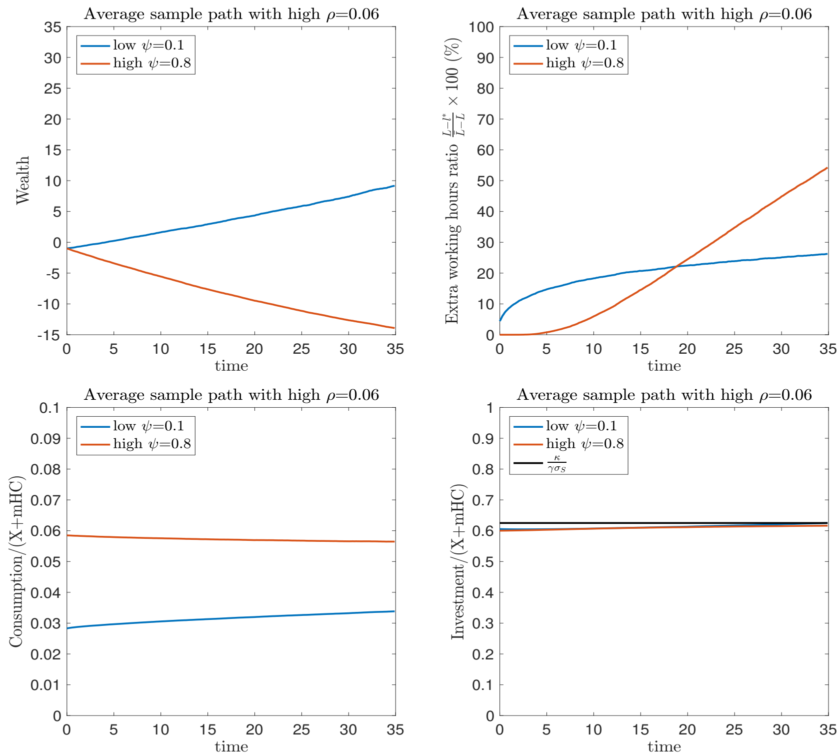

Figure 8: Average sample path over 35 years when the agent is impatient ($\rho = 0.06$). The other parameters are the same as those in Figure 2.

the average values of the entire sample paths. However, considering only the average values might sometimes be misleading. More precisely, any type of agent will be able to enjoy the maximum leisure if he/she is lucky, i.e., he/she receives a lot of good shocks for a long time and thus becomes richer and richer. On the other hand, any type of agent should do work more and more if he/she is unlucky in that he/she receives a lot of bad shocks for a long time and thus becomes poorer and poorer. If the average working hours of this unlucky group is very high (low), it can significantly contribute to the increase (reduction) in the average. In fact, the average working hours of Type Y agents gradually increase, which is driven in fact by the extremely unlucky group, as will be seen below.

Figure 9 shows the dynamics of wealth and the extra working hours by each subgroup. More precisely, we divide the 10,000 samples into five quintiles: top 20%, top 20-40%, top 40-60%, top 60-80%, and bottom 20% with respect to the final wealth. Then, we exclude the top and bottom 20% and plot the average wealth and the ratio of the average extra working hours to the minimum working hours of the middle three groups in the sample paths. The wealth level of the Type N_e agent is the highest, that of the Type N agent is the second, that of the Type Y agent is the third, and that of the Type Y_e agent is the lowest for the entire life for every subgroup sample. The Type N_e agent's extra working hours are the highest, the Type N agent's are the second, and the Type Y agents are the lowest for most of lifetime if we compare only the three groups: N_e, N, and Y. However, the extra working hours of Y_e are very different: they are the lowest in early life and become the highest in later life. For all subgroups, the wealth level of the Type Y_e decreases and the extra working hours of the Type Y_e increase over time.

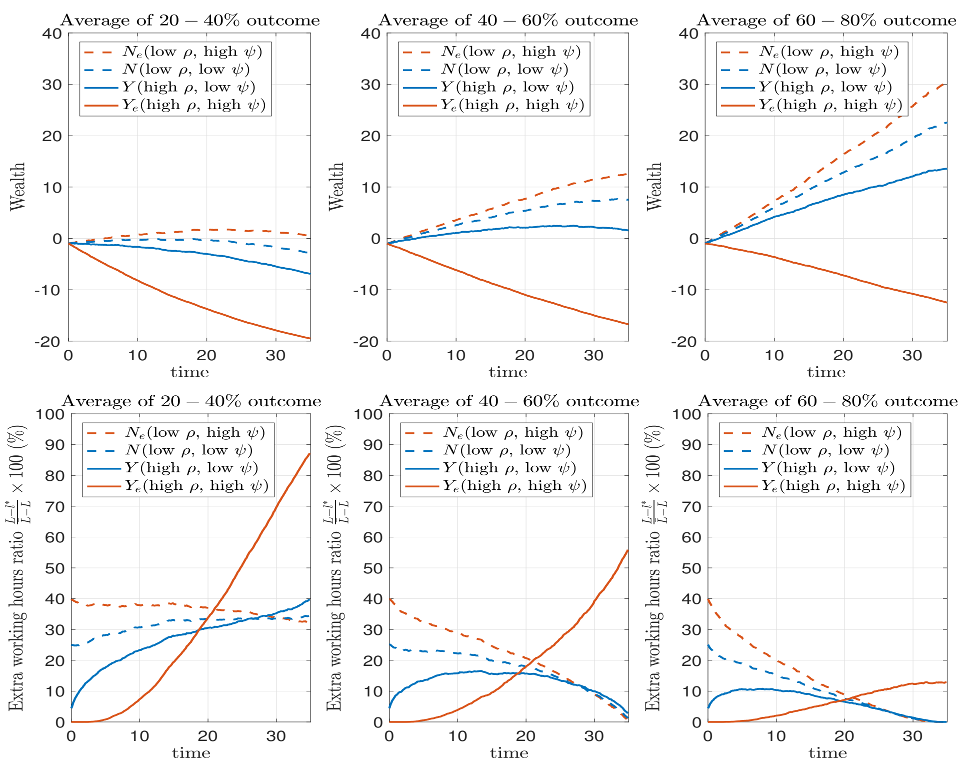

Figure 9: The average wealth and extra working hours of three quintiles (top 20-40% (left panel), top 40-60% (middle panel), and top 60-80% (right panel) of the total samples in the final wealth level).

In summary, going back to the classification summarized in Table 1, we say that a life pattern is sustainable if the agent, when old, is likely to maintain the same leisure pattern as when they were young. Note that the ratio of consumption to the (minimum) human wealth is stable for every type of agent. In this sense, the consumption choice is consistent for any type of agent. Therefore, we determine sustainability by using the long-run consistency of the labor-leisure choice.

There are two types of individuals who choose YOLO (Type Y and Y_e). YOLO is sustainable for Type Y agents in the sense that they are likely to maintain their working hours steadily in early and later life, especially, as shown by the dynamics of wealth and the working hours by each subgroup in Figure 9.[14] The speed of their wealth accumulation is low, but wealth gradually tends to increase. In contrast, if the YOLO style results from the preference of being simply impatient and more eager to substitute current consumption with future income, the individual with such preference (i.e., Type Y_e) will work longer and longer when old, while he or she can enjoy much leisure when young. This life pattern indeed contradicts the spirit of YOLO: we regard it as unsustainable YOLO.

5 Wealth Effect and Policy Implications

We presented the long-run sustainability analysis in Section 4.2 based on the simulation results. Note that in the simulations, we let each agent start with the same initial wealth level and track each agent's consumption and labor-leisure choices in time.

In this section, we demonstrate that individuals' types evolve over time due to the influence of the wealth effect. Specifically, an individual's decisions regarding work and leisure can significantly vary depending on their wealth status. For instance, it is possible that an individual who was previously inclined towards a financially prudent (or sustainable) lifestyle (i.e., characterized as Y, N, and N_e types) can transition to an extreme YOLO type (Y_e) if they experience financial setbacks over time. In our context, it is important to note that these setbacks are not necessarily idiosyncratic, but can result from systemic shocks in the financial market. Therefore, this finding carries significant implications for policymaking, which will be discussed in Section 6.

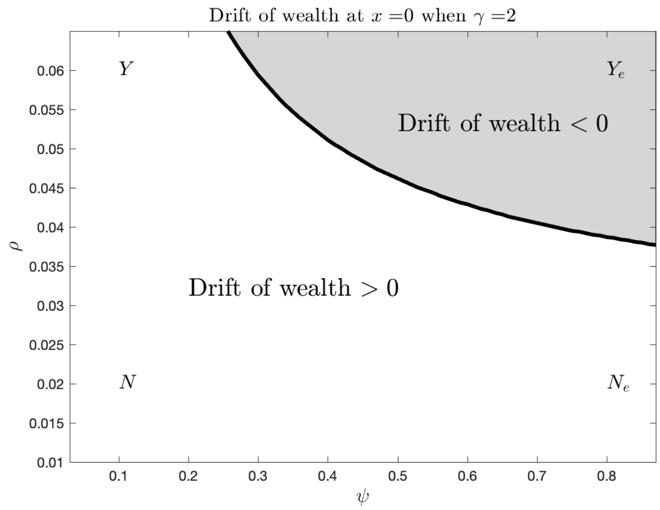

Figure 10: The sign of $D(x)$ when $x = 0$ in the (ψ, ρ)-plane.

5.1 Wealth Effect on Life Styles

First, we start with the static but more analytic characterization beyond the sustainability results. We visualize the life patterns in the (ψ, ρ)-plane by using the expected wealth change for each x. In this case, the expected change in wealth at $X_t = x$ or the drift of the optimal wealth process from (2.3) is

$$D(x) \triangleq rx + (\mu_S - r)\pi^*(x) - c^*(x) + w(\bar{L} - l^*(x)), \tag{5.1}$$

where $\pi^*(x)$, $c^*(x)$, and $l^*(x)$ are given by (3.17), (3.18), and (3.19), respectively. Figure 10 illustrates two regions in which $D(x)$ have different signs for $x = 0$ on the (ψ, ρ)-plane. In particular, the gray region represents the set of $\{(\psi, \rho) \,|\, D(0) < 0\}$. Each type of agent is located near the corner of the (ψ, ρ)-plane, as specified in the figure. We can easily see that the Y_e life pattern is not sustainable since its expected wealth change is fairly negative ($\mathbb{E}[dX|X = 0] << 0$). The drift of N_e is the highest, that of N is the second, that of Y is the third, and that of Y_e is the lowest. Note that it is a clockwise order: the highest at the right-bottom corner and the lowest at the right-top corner. The reason is that the impact of EIS is opposite for the cases of high discounting and low discounting. The agent with (ψ, ρ) closer

[14]More precisely, the working hours of the top 40-80% decrease later in life and they outweigh the increasing working hours of the top 20-40%.

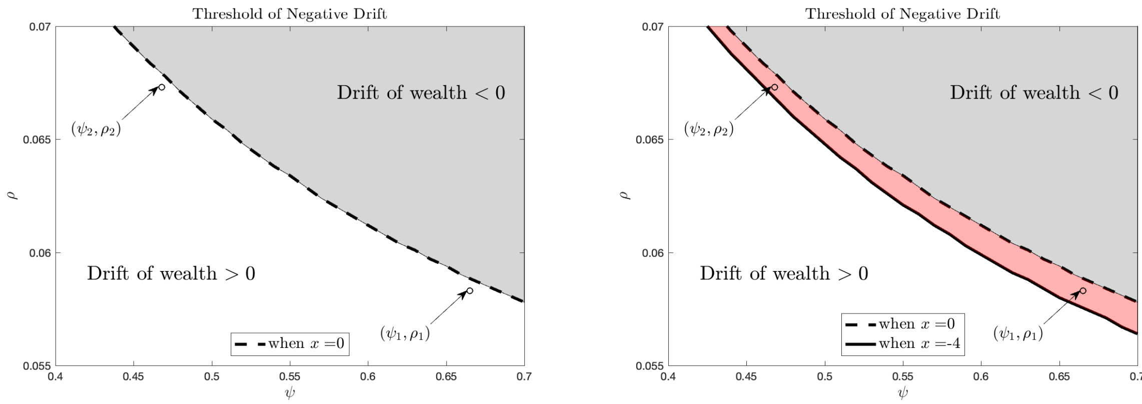

Figure 11: The region where $D(x) < 0$ expands as x decreases. The grey area in the left panel is the region of $D(x) < 0$ when $x = 0$ and the entire colored area including the grey area in the right panel is the region of $D(x) < 0$ when when $x = -4$. (ψ_1, ρ_1) is a Type N_e agent and (ψ_2, ρ_2) is a Type Y agent in the left panel, but both becomes type Y_e in the right panel after getting bad financial shocks.

to the top-right corner is more likely to have unsustainable YOLO as this agent's wealth in the next period is more likely be smaller than current wealth. In contrast, the agent with (ψ, ρ) closer to the top-left corner is more likely to have sustainable YOLO.

Second and more important, we highlight that the extent of the gray area increases as wealth decreases as illustrated in Figure 11. This observation suggests that an individual's lifestyle can evolve over time. To be more precise, whether someone embraces a YOLO lifestyle or not can significantly hinge on luck over time. To see the impact of wealth, for instance, consider two agents in Figure 11 indexed by a preference pair (ψ_i, ρ_i) at time t for $i = 1, 2$. Note that both (ψ_i, ρ_i)'s fall outside the shaded region depicted in the left panel of Figure 11: The current classification of Agent i is N_e and Y for $i = 1, 2$, respectively. However, if the agents experience a significant negative financial shock during $(t, t+h)$, the shaded region expands, causing (ψ_i, ρ_i) to be situated within it at time $t + h$ as in the right panel of Figure 11.

These transitions imply that a large financial setback can lead to a dramatic reduction in working hours for many individuals. It might not be striking that the transition from Y to Y_e thank to the bad shock for agent 2. However, the case of Agent 1 demonstrates that even an individual who was initially classified as an extreme non-YOLO type can transition to being an extreme YOLO type some time later. Conversely, an agent whose life style is an extreme non-YOLO at some point of time can become an extreme YOLO type after experiencing significant bad shocks on his/her financial status. This shift between Y_e and N_e is a dramatic change in terms of the life pattern choice, which highlights the importance of the financial status when the agent chooses a life pattern. This finding leads to policy implications that will be discussed in Section 6.

5.2 Role of EIS on Sustainability

Related to the graphical presentation, Proposition 2 provides an important implication for the role of EIS in determining the sustainability of a life pattern.

Proposition 2. *When $\theta = 1$ and the agent is impatient enough so that*

$$\rho > \hat{\rho} \triangleq r + \frac{\kappa^2}{2} \max \left[\frac{\gamma + 1}{\gamma}, \frac{\gamma_2 + 1}{\gamma_2} \right], \tag{5.2}$$

the drift of the optimal wealth process is negative, that is, $D(x) < 0$, for all x.

Proposition 2 implies that in the case of the additive-separable utility setup, the life pattern of an agent with high ρ is unlikely to be sustainable. More precisely, there exists a cutoff $\hat{\rho}$ such that if the agent's time preference is greater than $\hat{\rho}$, his/her life pattern is unsustainable for any given wealth. For example, in Figure 10 all the agents with $\rho > \hat{\rho}$ are extreme YOLO types for the case of the CRRA utility function. This result is consistent with that from the consumption sustainability literature (Campbell and Martin (2022); Campbell and Sigalov (2022); Pindyck (2022)).

However, we find that in a general recursive utility setup, the life pattern chosen by the agent with high subject discounting can be sustainable if the agent's EIS is low (i.e., the case for Type Y agents). The agent's with (ψ, ρ) closer top-left corner is of type Y and his/her life pattern is more likely to be sustainable. In addition, the key point is that the grey area decreases with wealth. In other words, the agent with any pair of (ψ, ρ) optimally chooses a sustainable life pattern when his/her wealth becomes sufficiently high. This observation leads to policy implications in the next subsection.

6 Policy Implications

Before we delve into police suggestions, let us first address a fundamental question: why is YOLO so popular, especially among Gen Z and millennials? It is essential to note that there is no substantial evidence indicating a distinct preference distribution among these younger generations compared to older ones (Weeks and Schaffert (2019), Zabel et al. (2017), and Pyoria et al. (2017)). In other words, one cannot argue that YOLO became popular merely because these generations were inherently predisposed to it. We acknowledge the possibility of behavioral explanations rooted in societal or environmental factors that influenced the younger generations during their formative years. However, the primary objective of this paper is to propose economic policy implications based on the rational expectation framework.

It is crucial to highlight the significant disparity in financial situations between the current young generations and their predecessors when they enter the workforce after graduating. Specifically, Gen Z and millennials are substantially burdened by high student loan debts and soaring housing prices, as pointed out in studies such as (FINESSE, 2017). Numerous reports have underscored the alarming trend that the current young generation is likely to be financially disadvantaged compared to their parents over their lifetime, primarily due to these challenges. Moreover, the lack of robust retirement savings provided by employers, as indicated by (Brown, 2018), further exacerbates their financial constraints in early adulthood.

Considering the aforementioned factors, it becomes apparent within our theoretical framework that the popularity of YOLO among the current young generation can be attributed to their challenging financial circumstances, both in absolute terms (such as high levels of debt and inadequate retirement savings) and relative terms (including significantly higher housing costs and a seemingly diminished lifetime income compared to previous generations). For example, according to the Consumer Credit Data (G.19) by the federal Reserve System, the total student loan debt has almost tripled over 15 years and Federal student debt has grown by over 8.5 times since 1995 (extensive margin)[15] Adjusting inflation, the average amount of debt of college graduates per borrower are \$14,061 in 1990, but it increases to \$31,500 in 2020 (intensive margin). While the absolute debt growth rate is 124%, the income growth rate is much smaller. More precisely, the average debt-to-income ratio defined by the debt

[15] See the summary in `https://tinyurl.com/5n8r89ww`.

at college graduation to the average starting salary is 24.8% in 1990 and 54.3% in 2020[16]. The average sales price of houses sold for the United States is $150,100 in Q4 of 1990 and $403,900 in Q4 of 2020 (FRED Economic Data).[17]

To illustrate the economic environments graphically, an individual born 2-30 years ago is far more likely to fall within the shaded region in Figure 11 than an individual born 4-60 years ago, given that they are given the same preference parameters. This disparity vividly illustrates the financial constraints faced by the current young generation, providing a compelling explanation for the widespread adoption of the YOLO mentality among them.

Therefore, we argue that any polices that directly or indirectly improve the financial status of the current young will effectively encourage the young to choose sustainable life styles. A notable example is the substantial reduction or exemption of student loans. Another approach involves providing significant tax credits for first-time home buyers within this age group and offering employers subsidies to increase employees' retirement savings. However, implementing these measures can be financially burdensome. While it is beyond the scope of the current paper, we can cautiously suggest a heuristic idea for the cost-benefit analysis. There needs to be a balance between the cost of distributing these subsidies and the benefits gained from encouraging a substantial number of individuals, who might otherwise adopt extreme YOLO life styles, to embrace more sustainable lifestyles. Considering this trade-off, we can identify an optimal subsidy policy that maximizes social welfare.

7 Concluding Remarks

In this paper we set up a model of a rational economic agent with recursive utility and a flexible labor-leisure choice with the minimum working hour requirement. We characterize the types of agents by using subject discounting and EIS in terms of their consumption and leisure patterns. We also examined whether these lifestyles are sustainable in the long-run, i.e., whether agents, when old, can continue to have the same lifestyle as they had when they were young. We also found that lifestyles are time-varying according to the agent's financial status as well as the agent's preference. Based on this fact, we suggest several policy

[16] https://wordsrated.com/student-loan-debt-by-year/
[17] https://fred.stlouisfed.org/series/ASPUS

implications to address the long-term challenges that can arise from discouraged workers transitioning from sustainable lifestyles to unsustainable ones due to financial setbacks.

Our model is fairly stylized and stringent in the sense that the model abstracts from two important aspects of the labor-leisure choice in real life: we did not consider *voluntary retirement* and we assume that the wage rate is constant in our model. With respect to the first, adding an optimal retirement decision into the agent's action would not change the implications much. For instance, we show that the lifestyle of the Type Y_e agent is not sustainable because he/she becomes poorer on average later in life. It is well-known from the retirement literature that an economic agent optimally retires if and only if he/she becomes sufficiently rich (Choi and Shim (2006), Farhi and Panageas (2007), and Choi et al. (2008)). In our case, Type Y_e agents will not be able to retire early since they will become poorer over time. They also will work more later in life. Therefore, it will still be true that under the voluntary retirement setup, the YOLO-like lifestyle of Type Y_e agents is not sustainable and they can hardly retire.

Regarding the wage, adding uncertainty or risk in the wage rate into our model would be very interesting. It would change the labor-leisure choice over the time, which could provide various additional implications. While the analysis will be fairly complicated (perhaps no analytic solutions would be obtained), an intuitive conjecture is that YOLO type agents would enjoy more leisure during the time when the wage rate is high. The expected growth rate of wealth would be different depending on the volatility of the wage rate dynamics. However, we doubt that the result for long-term sustainability will be different from the case in which the wage rate is constant.

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Appendix

The Appendix consists of four parts. Appendix A provides the discussion on the aggregator and the assumption of $\theta > 0$. We present our solution method for $\varphi(z)$ in Section A. Section B provide all the proofs for the propositions and the corollaries in the main body of the paper. Finally, the numerical scheme is explained in Section C.

A Discussion on the aggregator and the assumption $\theta > 0$

Herdegen et al. (2023a) provide a thorough discussion on the difference between the following two types of aggregators[18],

$$g_{EZ}(t,c,V) = be^{-\rho t}\frac{c^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}}((1-\gamma)V)^{\frac{\frac{1}{\psi}-\gamma}{1-\gamma}}, \tag{A.1}$$

$$g_{EZ}^{\Delta}(c,V) = b\frac{c^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}}((1-\gamma)V)^{\frac{\frac{1}{\psi}-\gamma}{1-\gamma}} - \frac{\rho}{\theta}V. \tag{A.2}$$

As in Herdegen et al. (2023a), we refer to the aggregators in (A.1) and (A.2) as the *discounted* form and *difference* form, respectively.

Although the difference form aggregator in (A.2) has been widely used in the literature, the discounted form aggregator in (A.1), which is a reformulation of the discounted form in (A.1), has several advantages. Among them, the most important advantage of using discounted form aggregator in (A.1) is that the existence of utility process V for a given consumption process c is guaranteed for more wide class of consumption processes. More specifically, if there exists a utility process V for a given consumption process c with difference form aggregator, same V is also the utility process with discounted form aggregator. However, the converse may not be true.

It is shown in Herdegen et al. (2023a) that $\vartheta > 0$ ($\theta > 0$ in our model) is necessary for the existence of utility process V for a given consumption process c. More detailed discussion

[18] γ, ψ, θ, and ρ in our model correspond to R,$1/S$,$1/\vartheta$, and δ in Herdegen et al. (2023a)

and analysis on the existence and uniqueness of utility process V for a given consumption process c are provided in Herdegen et al. (2023b) (when $0 < \vartheta < 1$, which is $\theta > 1$ in our model) and Herdegen et al. (2021) (when $\vartheta > 1$, which is $0 < \theta < 1$ in our model).

Based on these results by Herdegen et al. (2023a,b, 2021), we adopt the discounted form aggregator in (2.2) by setting[19]

$$f(s, c, l, V) = g_{EZ}(s, c^{\eta} l^{1-\eta}, V)$$

and assume that $\theta > 0$ to guarantee the existence and uniqueness of utility process V for given consumption process c and the leisure process l. Since both c and l are nonnegative progressively measurable processes, $c^{\eta} l^{1-\eta}$ is also a nonnegative progressively measurable process. Thus, the results by Herdegen et al. (2023a,b, 2021) on the existence and uniqueness of utility process V can be applied to our model with discounted form aggregator in (2.2).

[19]Since b in (A.1) is just a scaling parameter for the utility process V, which is irrelevant to the preference and the optimal decision of agent, we assume that $b = 1$ without loss of generality.

Online Appendix for "Consumption and Labor-Leisure Choices: YOLO and Long-Run Sustainability"

K.J Choi, M. Kwak, and B.H. Lim

A Solution Analysis for $\varphi(z)$

As we will see in the proof of Proposition A.1, by representing $\alpha_1(z)$, $\beta_1(z)$, $\alpha_2(z)$, and $\beta_2(z)$ in terms of integrals of $\varphi_1(z)$ and $\varphi_2(z)$, we can derive the coupled integral equations of $\varphi_1(z)$ and $\varphi_2(z)$ that include the free boundary $\bar{z}$, and an algebraic equation for the free boundary $\bar{z}$ that includes the integrals of $\varphi_1(z)$ and $\varphi_2(z)$. However, since the equations are complicated and highly interconnected, it is challenging to obtain the solutions directly. Thus, we take an alternative way to obtain the solution by introducing another transform.

Let us define a new variable $y = z/\bar{z}$, and define new functions $F_1(y)$ and $F_2(y)$ as follows:

$$F_1(y) = \frac{(1-\gamma)(\varphi_1(z) - z\varphi_1'(z))}{z^{1-\frac{1}{\gamma}}}, \quad \text{for } y \geq 1, \tag{A.1}$$

$$F_2(y) = \frac{(1-\gamma)(\varphi_2(z) - z\varphi_2'(z))}{z^{1-\frac{1}{\gamma_2}}}, \quad \text{for } 0 < y \leq 1. \tag{A.2}$$

Then, we represent $\alpha_1(z)$, $\beta_1(z)$, $\alpha_2(z)$, and $\beta_2(z)$ in terms of $F_1(y)$ and $F_2(y)$. Consequently, we can derive the integral equations for $F_1(y)$ and $F_2(y)$. Although the integral equations of $F_1(y)$ and $F_2(y)$ are still interconnected, they are much simpler than the integral equations of $\varphi_1(z)$ and $\varphi_2(z)$, and more importantly, they do not include the free boundary $\bar{z}$. Thus, we will solve $F_1(y)$ and $F_2(y)$ without determining the free boundary $\bar{z}$ simultaneously. Once we obtain $F_1(1)$, the free boundary $\bar{z}$ is explicitly determined. The following proposition summarizes the above results and provides the solutions to ODEs (3.9) and (3.10).

Proposition A.1. *Let us define M_1, M_2, ζ as*

$$M_1 \triangleq \frac{(n_+ - 1 + \frac{1}{\gamma})(1 - n_- - \frac{1}{\gamma})}{(n_+ - n_-)(1-\gamma)} \gamma A_1^{1+(\theta-1)\psi}, \tag{A.3}$$

$$M_2 \triangleq \frac{(n_+ - 1 + \frac{1}{\gamma_2})(1 - n_- - \frac{1}{\gamma_2})}{(n_+ - n_-)(1-\gamma)} \gamma_2 \left(\frac{A_2}{\eta}\right)^{1+\frac{\theta-1}{\gamma_1}}, \tag{A.4}$$

$$\zeta \triangleq \eta \left(\frac{\eta w}{1-\eta}\right)^{-\gamma_1} L^{-\frac{1}{\psi}}. \tag{A.5}$$

Then, the solutions to the nonlinear ODEs in (3.9) *and* (3.10) *are given by* (3.14) *and* (3.15)*, respectively, with*

$$\begin{aligned}
\alpha_1(z) =& M_1 \bar{z}^{-(n_+ - 1 + \frac{1}{\gamma})} \int_{z/\bar{z}}^{\infty} u^{-n_+ - \frac{1}{\gamma}} F_1(u)^{-(\theta-1)\psi} du - M_1 \frac{A_1^{-(\theta-1)\psi}}{(n_+ - 1 + \frac{1}{\gamma})} z^{-(n_+ - 1 + \frac{1}{\gamma})}, \\
\beta_1(z) =& M_1 \frac{(n_+ - 1)}{(1 - n_-)} \bar{z}^{1 - n_- - \frac{1}{\gamma}} \int_1^{\infty} u^{-n_+ - \frac{1}{\gamma}} F_1(u)^{-(\theta-1)\psi} du + M_1 \bar{z}^{1 - n_- - \frac{1}{\gamma}} \int_1^{z/\bar{z}} u^{-n_- - \frac{1}{\gamma}} F_1(u)^{-(\theta-1)\psi} du \\
&+ \frac{\zeta^{\frac{1}{\theta-1}}}{(1-n_-)(1-\gamma)} \bar{z}^{\frac{1}{1-\theta} - n_-} - M_1 \frac{A_1^{-(\theta-1)\psi}}{(1 - n_- - \frac{1}{\gamma})} z^{1 - n_- - \frac{1}{\gamma}},
\end{aligned}$$

$$\begin{aligned}
\alpha_2(z) =& M_1 \bar{z}^{-(n_+ - 1 + \frac{1}{\gamma})} \int_1^{\infty} u^{-n_+ - \frac{1}{\gamma}} F_1(u)^{-(\theta-1)\psi} + M_2 \bar{z}^{-(n_+ - 1 + \frac{1}{\gamma_2})} \int_{z/\bar{z}}^{1} u^{-n_+ - \frac{1}{\gamma_2}} F_2(u)^{-\frac{\theta-1}{\gamma_1}} du \\
&+ \frac{(1-n_-) w L}{(n_+ - n_-) r} \bar{z}^{-(n_+ - 1)} - M_2 \frac{\left(\frac{A_2}{\eta}\right)^{-\frac{\theta-1}{\gamma_1}}}{(n_+ - 1 + \frac{1}{\gamma_2})} z^{-(n_+ - 1 + \frac{1}{\gamma_2})}, \\
\beta_2(z) =& M_2 \bar{z}^{1 - n_- - \frac{1}{\gamma_2}} \int_0^{z/\bar{z}} u^{-n_- - \frac{1}{\gamma_2}} F_2(u)^{-\frac{\theta-1}{\gamma_1}} - M_2 \frac{\left(\frac{A_2}{\eta}\right)^{-\frac{\theta-1}{\gamma_1}}}{(1 - n_- - \frac{1}{\gamma_2})} z^{1 - n_- - \frac{1}{\gamma_2}},
\end{aligned}$$

and the free boundary $\bar{z}$ is determined as

$$\bar{z} = F_1(1)^{-\gamma\psi(\theta-1)} \zeta^{\psi\gamma}. \tag{A.6}$$

The integral equations for $F_1(y)$, $F_2(y)$, and $F_1(1)$ are provided in the proof.

Before we provide the proof of the proposition, we have an important remark.

Remark 3. *Since $\varphi_1(z)$ and $\varphi_2(z)$ do not appear in $\alpha_1(z)$, $\beta_1(z)$, $\alpha_2(z)$, and $\beta_2(z)$ in Proposition A.1, one may think that we can define $F_1(y)$ and $F_2(y)$ directly from the value* *function $V(x)$ without defining the functions $\varphi_1(z)$ and $\varphi_2(z)$. Note that, however, the new variable $y = z/\bar{z}$ for $F_1(y)$ and $F_2(y)$ is defined using z and $\bar{z}$. Since the variable z is defined implicitly using $\varphi'(z)$ and $\bar{z}$ is the free boundary for $\varphi_1(z)$ and $\varphi_2(z)$, we have to define $\varphi_1(z)$, $\varphi_2(z)$, and the free boundary $\bar{z}$ before introducing $F_1(y)$ and $F_2(y)$ with new variable $y = z/\bar{z}$.*

Proof. We can show that

$$K = \frac{1}{2}\kappa^2(n_+ - 1 + \frac{1}{\gamma})(1 - n_- - \frac{1}{\gamma}), \tag{A.7}$$

$$K_2 = \frac{1}{2}\kappa^2(n_+ - 1 + \frac{1}{\gamma_2})(1 - n_- - \frac{1}{\gamma_2}). \tag{A.8}$$

Using A_1 in (3.13) and K in (A.7), we can show that M_1 defined in (A.3) satisfies

$$\frac{\eta^\psi}{\eta(\psi - 1)}\left(\frac{\eta w}{1-\eta}\right)^{1-\gamma_1\psi} = \frac{(n_+ - n_-)\kappa^2}{2}M_1. \tag{A.9}$$

Similarly, using A_2 in (3.13) and K_2 in (A.8), we can derive that M_2 in (A.4) satisfies

$$\frac{\gamma_1}{1-\gamma_1}\eta^{\frac{1}{\gamma_1}}L^{1-\frac{1}{\gamma_1\psi}} = \frac{(n_+ - n_-)\kappa^2}{2}M_2. \tag{A.10}$$

For $z > \bar{z}$, if we substitute $\varphi_1(z)$ in (3.14) into the ODE (3.9) and using (A.9), we have the following equation:

$$\frac{\alpha_1'(z)n_+z^{n_++1} + \beta_1'(z)n_-z^{n_-+1}}{(n_+ - n_-)} = -M_1\left[((1-\gamma)(\varphi_1(z) - z\varphi_1'(z))^{-(\theta-1)\psi}z^{1-\psi} - A_1^{-(\theta-1)\psi}z^{1-\frac{1}{\gamma}}\right]. \tag{A.11}$$

From (3.16), we have

$$\alpha_1'(z)n_+z^{n_++1} + \beta_1'(z)n_-z^{n_-+1} = \alpha_1'(z)(n_+ - n_-)z^{n_++1} = -\beta_1'(z)(n_+ - n_-)z^{n_-+1}.$$

Thus, we have two different versions of (A.11) as follows:

$$\alpha_1'(\xi) = -M_1\left[((1-\gamma)(\varphi_1(\xi)-\xi\varphi_1'(\xi)))^{-(\theta-1)\psi}\xi^{-n_+-\psi} - A_1^{-(\theta-1)\psi}\xi^{-n_+-\frac{1}{\gamma}}\right], \tag{A.12}$$

$$\beta_1'(\xi) = M_1\left[((1-\gamma)(\varphi_1(\xi)-\xi\varphi_1'(\xi)))^{-(\theta-1)\psi}\xi^{-n_--\psi} - A_1^{-(\theta-1)\psi}\xi^{-n_--\frac{1}{\gamma}}\right], \tag{A.13}$$

for $\xi > \bar{z}$. For given $z > \bar{z}$, by integrating (A.12) from z to ∞ and setting $\lim_{z\to\infty}\alpha_1(z) = 0$ due to the growth condition, we have

$$\alpha_1(z) = M_1\left[\int_z^\infty ((1-\gamma)(\varphi_1(\xi)-\xi\varphi_1'(\xi)))^{-(\theta-1)\psi}\xi^{-n_+-\psi}d\xi - \int_z^\infty A_1^{-(\theta-1)\psi}\xi^{-n_+-\frac{1}{\gamma}}d\xi\right]. \tag{A.14}$$

By integrating (A.13) from $\bar{z}$ to z, we have

$$\beta_1(z) = \beta_1(\bar{z}) + M_1\left[\int_{\bar{z}}^z ((1-\gamma)(\varphi_1(\xi)-\xi\varphi_1'(\xi)))^{-(\theta-1)\psi}\xi^{-n_--\psi}d\xi - \int_{\bar{z}}^z A_1^{-(\theta-1)\psi}\xi^{-n_--\frac{1}{\gamma}}d\xi\right]. \tag{A.15}$$

For $0 < z \leq \bar{z}$, by substituting $\varphi_2(z)$ in (3.15) into the ODE (3.10) and using the relationship (A.10), we have

$$\frac{\alpha_2'(z)n_+z^{n_++1} + \beta_2'(z)n_-z^{n_-+1}}{(n_+-n_-)} = -M_2\left[((1-\gamma)(\varphi_2(z)-z\varphi_2'(z))^{-\frac{\theta-1}{\gamma_1}}z^{1-\frac{1}{\gamma_1}} - \left(\frac{A_2}{\eta}\right)^{-\frac{\theta-1}{\gamma_1}}z^{1-\frac{1}{\gamma_2}}\right]. \tag{A.16}$$

Again, from (3.16), we have

$$\alpha_2'(z)n_+z^{n_++1} + \beta_2'(z)n_-z^{n_-+1} = \alpha_2'(z)(n_+-n_-)z^{n_++1} = -\beta_2'(z)(n_+-n_-)z^{n_-+1}.$$

Therefore, we have two different versions of (A.16) as follows:

$$\alpha_2'(\xi) = -M_2\left[((1-\gamma)(\varphi_2(\xi)-\xi\varphi_2'(\xi))^{-\frac{\theta-1}{\gamma_1}}\xi^{-n_+-\frac{1}{\gamma_1}} - \left(\frac{A_2}{\eta}\right)^{-\frac{\theta-1}{\gamma_1}}\xi^{-n_+-\frac{1}{\gamma_2}}\right], \tag{A.17}$$

$$\beta_2'(\xi) = M_2\left[((1-\gamma)(\varphi_2(\xi)-\xi\varphi_2'(\xi))^{-\frac{\theta-1}{\gamma_1}}\xi^{-n_--\frac{1}{\gamma_1}} - \left(\frac{A_2}{\eta}\right)^{-\frac{\theta-1}{\gamma_1}}\xi^{-n_--\frac{1}{\gamma_2}}\right], \tag{A.18}$$

for $0 < \xi \leq \bar{z}$. For given $0 < z \leq \bar{z}$, by integrating (A.17) from z to $\bar{z}$, we have

$$\alpha_2(z) = \alpha_2(\bar{z}) + M_2\left[\int_z^{\bar{z}}((1-\gamma)(\varphi_2(\xi)-\xi\varphi_2'(\xi))^{-\frac{\theta-1}{\gamma_1}}\xi^{-n_+-\frac{1}{\gamma_1}}d\xi - \int_z^{\bar{z}}\left(\frac{A_2}{\eta}\right)^{-\frac{\theta-1}{\gamma_1}}\xi^{-n_+-\frac{1}{\gamma_2}}d\xi\right]. \tag{A.19}$$

By integrating (A.18) from 0 to z and setting $\lim_{z\to 0+}\beta_2(z) = 0$ because of the growth condition, we get

$$\beta_2(z) = M_2\left[\int_0^{z}((1-\gamma)(\varphi_2(\xi)-\xi\varphi_2'(\xi))^{-\frac{\theta-1}{\gamma_1}}\xi^{-n_--\frac{1}{\gamma_1}}d\xi - \int_0^{z}\left(\frac{A_2}{\eta}\right)^{-\frac{\theta-1}{\gamma_1}}\xi^{-n_--\frac{1}{\gamma_2}}d\xi\right]. \tag{A.20}$$

Note that we have three unknown values: $\bar{z}$, $\beta_1(\bar{z})$, and $\alpha_2(\bar{z})$. These unknown values can be determined by using the smooth-pasting conditions (3.11) at $z = \bar{z}$, and setting $l^*(\bar{x}) = L$ for $l^*(x)$ given in (3.2). First, the smooth-pasting conditions (3.11) at $z = \bar{z}$ lead to

$$\begin{aligned}
&\alpha_1(\bar{z})\bar{z}^{n_+} + \beta_1(\bar{z})\bar{z}^{n_-} + \varphi_{p,1}(\bar{z}) = \alpha_2(\bar{z})\bar{z}^{n_+} + \beta_2(\bar{z})\bar{z}^{n_-} + \varphi_{p,2}(\bar{z}),\\
&\alpha_1(\bar{z})n_+\bar{z}^{n_+-1} + \beta_1(\bar{z})n_-\bar{z}^{n_--1} + \varphi_{p,1}'(\bar{z}) = \alpha_2(\bar{z})n_+\bar{z}^{n_+-1} + \beta_2(\bar{z})n_-\bar{z}^{n_--1} + \varphi_{p,2}'(\bar{z}),
\end{aligned}$$

and combining above two equations, we can obtain

$$\beta_1(\bar{z}) = \beta_2(\bar{z}) + \frac{\bar{z}^{-n_-}}{(n_+ - n_-)} \left[-\varphi_{p,1}(\bar{z})n_+ + \bar{z}\varphi'_{p,1}(\bar{z}) + \varphi_{p,2}(\bar{z})n_+ - \bar{z}\varphi'_{p,2}(\bar{z})\right], \tag{A.21}$$

$$\alpha_2(\bar{z}) = \alpha_1(\bar{z}) + \frac{\bar{z}^{-n_+}}{(n_+ - n_-)} \left[-\varphi_{p,1}(\bar{z})n_- + \bar{z}\varphi'_{p,1}(\bar{z}) + \varphi_{p,2}(\bar{z})n_- - \bar{z}\varphi'_{p,2}(\bar{z})\right]. \tag{A.22}$$

Although we obtain expressions for $\beta_1(\bar{z})$ and $\alpha_2(\beta z)$, we still need one more equation to determine $\bar{z}$. We can show that $l^*(\bar{x}) = L$ is equivalent to

$$(1-\gamma)(\varphi_1(\bar{z}) - \bar{z}\varphi'_1(\bar{z})) = \zeta^{\frac{1}{\theta-1}}\bar{z}^{\frac{1}{1-\theta}}, \tag{A.23}$$

where ζ is given in (A.5). From (A.23), we can derive another expression of $\beta_1(\bar{z})$ as follows:

$$\beta_1(\bar{z}) = \frac{(n_+-1)}{(1-n_-)}\alpha_1(\bar{z})\bar{z}^{n_+-n_-} - \frac{(\varphi_{p,1}(\bar{z}) - \bar{z}\varphi'_{p,1}(\bar{z}))}{(1-n_-)(1-\gamma)} + \frac{\zeta^{\frac{1}{\theta-1}}\bar{z}^{\frac{1}{1-\theta}}}{(1-n_-)(1-\gamma)}. \tag{A.24}$$

Combining (A.21) and (A.24), we can obtain another equation as follows that determines $\bar{z}$:

$$\begin{aligned}
&\beta_2(\bar{z}) + \frac{\bar{z}^{-n_-}}{(n_+ - n_-)} \left[-\varphi_{p,1}(\bar{z})n_+ + \bar{z}\varphi'_{p,1}(\bar{z}) + \varphi_{p,2}(\bar{z})n_+ - \bar{z}\varphi'_{p,2}(\bar{z})\right] \\
&= \frac{(n_+-1)}{(1-n_-)}\alpha_1(\bar{z})\bar{z}^{n_+-n_-} - \frac{(\varphi_{p,1}(\bar{z}) - \bar{z}\varphi'_{p,1}(\bar{z}))}{(1-n_-)(1-\gamma)} + \frac{\zeta^{\frac{1}{\theta-1}}\bar{z}^{\frac{1}{1-\theta}}}{(1-n_-)(1-\gamma)}.
\end{aligned} \tag{A.25}$$

In summary, we have coupled integral equations for the functions $\varphi_1(z)$ and $\varphi_2(z)$ as follows

$$\varphi_1(z) = \alpha_1(z)z^{n_+} + \beta_1 z^{n_-} + \varphi_{p,1}(z), \quad \varphi_2(z) = \alpha_2(z)z^{n_+} + \beta_2 z^{n_-} + \varphi_{p,2}(z),$$

with $\alpha_1(z)$, $\beta_1(z)$, $\alpha_2(z)$, and $\beta_2(z)$ in (A.14), (A.15), (A.19), and (A.20), respectively, and $\bar{z}$ is determined implicitly by (A.25).

Even though we have obtained the integral equations for $\varphi_1(z)$, $\varphi_2(z)$, and $\bar{z}$ by deriving the integral representations for $\alpha_1(z)$, $\beta_1(z)$, $\alpha_2(z)$, and $\beta_2(z)$ as in (A.14), (A.15), (A.19), and (A.20), respectively, it is very complicated and interconnected each other. To resolve this, we introduce a new variable $y = z/\bar{z}$ and new functions $F_1(y)$ and $F_2(y)$ defined in

(A.1) and (A.2).

We can show that

$$
\begin{aligned}
&((1-\gamma)(\varphi_1(\xi)-\xi\varphi_1'(\xi)))^{-(\theta-1)\psi}\xi^{-n_\pm-\psi} = \xi^{-n_\pm-\frac{1}{\gamma}}F_1(\xi/\bar{z})^{-(\theta-1)\psi},\\
&((1-\gamma)(\varphi_2(\xi)-\xi\varphi_2'(\xi)))^{-\frac{\theta-1}{\gamma_1}}\xi^{-n_\pm-\frac{1}{\gamma_1}} = \xi^{-n_\pm-\frac{1}{\gamma_2}}F_2(\xi/\bar{z})^{-\frac{\theta-1}{\gamma_1}}.
\end{aligned}
$$

By applying above relationship to the equations (A.14), (A.15), (A.19), (A.20), (A.21), (A.22), (A.25), and changing the variable of integrals from ξ to $u = \xi/\bar{z}$, we can obtain the expressions for $\alpha_1(z)$, $\beta_1(z)$, $\alpha_2(z)$, $\beta_2(z)$ in Proposition A.1, and the integral equations for $F_1(y)$, $F_2(y)$, and $F_1(1)$ as follows:

$$
\begin{aligned}
F_1(y) = {} & F_1(1)y^{-(1-n_--\frac{1}{\gamma})} \qquad \text{(A.26)}\\
& -(n_+-1)M_1(1-\gamma)y^{n_+-1+\frac{1}{\gamma}}\int_y^\infty u^{-n_+-\frac{1}{\gamma}}F_1(u)^{-(\theta-1)\psi}du\\
& +(n_+-1)M_1(1-\gamma)y^{-(1-n_--\frac{1}{\gamma})}\int_1^\infty u^{-n_+-\frac{1}{\gamma}}F_1(u)^{-(\theta-1)\psi}du\\
& +(1-n_-)M_1(1-\gamma)y^{-(1-n_--\frac{1}{\gamma})}\int_1^y u^{-n_--\frac{1}{\gamma}}F_1(u)^{-(\theta-1)\psi}du, \quad \text{for } y \geq 1,
\end{aligned}
$$

$$
\begin{aligned}
F_2(y) = {} & F_2(1)y^{n_+-1+\frac{1}{\gamma_2}} \qquad \text{(A.27)}\\
& -(1-n_-)M_2(1-\gamma)y^{n_+-1+\frac{1}{\gamma_2}}\int_0^1 u^{-n_--\frac{1}{\gamma_2}}F_2(u)^{-\frac{\theta-1}{\gamma_1}}du\\
& -(n_+-1)M_2(1-\gamma)y^{n_+-1+\frac{1}{\gamma_2}}\int_y^1 u^{-n_+-\frac{1}{\gamma_2}}F_2(u)^{-\frac{\theta-1}{\gamma_1}}du\\
& +(1-n_-)M_2(1-\gamma)y^{-(1-n_--\frac{1}{\gamma_2})}\int_0^y u^{-n_--\frac{1}{\gamma_2}}F_2(u)^{-\frac{\theta-1}{\gamma_1}}du, \quad \text{for } 0 < y \leq 1,
\end{aligned}
$$

$$
\begin{aligned}
F_1(1) = {} & -(n_+-1)M_1(1-\gamma)\int_1^\infty u^{-n_+-\frac{1}{\gamma}}F_1(u)^{-(\theta-1)\psi}du\\
& +(1-n_-)M_2(1-\gamma)F_1(1)^{-\frac{(\gamma_2-\gamma)\psi(\theta-1)}{\gamma_2}}\zeta^{\frac{(\gamma_2-\gamma)\psi}{\gamma_2}}\int_0^1 u^{-n_--\frac{1}{\gamma_2}}F_2(u)^{-\frac{\theta-1}{\gamma_1}}du\\
& -\frac{(1-\gamma)(n_+-1)(1-n_-)}{(n_+-n_-)}\frac{wL}{r}F_1(1)^{-\psi(\theta-1)}\zeta^{\psi}, \qquad \text{(A.28)}
\end{aligned}
$$

with the free boundary $\bar{z}$ determined as (A.6). By the definition of $F_1(y)$ and $F_2(y)$ and the expression of $\bar{z}$ in (A.6), it is immediate that

$$F_2(1) = F_1(1)\bar{z}^{\frac{1}{\gamma_2}-\frac{1}{\gamma}} = F_1(1)^{\frac{\eta\gamma-\eta\psi\gamma+\psi\gamma}{\gamma_2}}\zeta^{\psi\gamma(\frac{1}{\gamma_2}-\frac{1}{\gamma})}. \tag{A.29}$$

$\square$

B Proofs

B.1 Proof of Proposition 1

By applying (3.7) and (3.8) to (3.2), (3.3), and (3.5), we can obtain the following candidates for the optimal strategies:

$$c^*(x) = \begin{cases} \frac{\eta w}{1-\eta} l^*(x), & x \le \bar{x}, \\ \left[\frac{L^{\psi^{-1}-\gamma_1}}{\eta}\left\{(1-\gamma)(\varphi_2(z) - z\varphi_2'(z))\right\}^{\theta-1} z\right]^{-\frac{1}{\gamma_1}}, & x \ge \bar{x}, \end{cases} \tag{B.1}$$

$$l^*(x) = \begin{cases} \eta^\psi \left(\frac{\eta w}{1-\eta}\right)^{-\gamma_1\psi}\left\{(1-\gamma)(\varphi_1(z) - z\varphi_1'(z))\right\}^{-(\theta-1)\psi} z^{-\psi}, & x \le \bar{x}, \\ L, & x \ge \bar{x}, \end{cases} \tag{B.2}$$

$$\pi^*(x) = \begin{cases} \frac{\kappa}{\sigma_S} z\varphi_1''(z), & x \le \bar{x}, \\ \frac{\kappa}{\sigma_S} z\varphi_2''(z), & x \ge \bar{x}, \end{cases} \tag{B.3}$$

where $x = -\varphi_1(z)$ for $x \le \bar{x}$ and $x = -\varphi_2(z)$ for $x \ge \bar{x}$. We can obtain the formulas in Proposition 1 by using $F_1(y)$, $F_2(y)$ defined in (A.1) and (A.2) along with $\bar{z}$ in (A.6) and $F_2(1)$ in (A.29).

B.2 Proof of Proposition 2

Assume that (5.2) holds. By substituting the optimal consumption, leisure, and investments in Corollary 1 into $D(x)$ in (5.1), we have

$$D(x) = \begin{cases} \left(r + \frac{\kappa^2}{\gamma} - K^{\theta=1}\right) A_1^{\theta=1}\mathcal{Z}_1^{\theta=1}(x)^{-\frac{1}{\gamma}} + \left\{-r + \kappa^2(n_- - 1)\right\} n_-\beta_1^{\theta=1}\mathcal{Z}_1^{\theta=1}(x)^{n_- -1}, & x \le \bar{x}^{\theta=1}, \\ \left(r + \frac{\kappa^2}{\gamma_2} - K_2^{\theta=1}\right) A_2^{\theta=1}\mathcal{Z}_2^{\theta=1}(x)^{-\frac{1}{\gamma_2}} + \left\{-r + \kappa^2(n_+ - 1)\right\} n_+\alpha_2^{\theta=1}\mathcal{Z}_2^{\theta=1}(x)^{n_+ -1}, & x \ge \bar{x}^{\theta=1}. \end{cases}$$

Since $n_- < 0$, it is obvious that $-r + \kappa^2(n_- - 1) < 0$. We also know from (3.20) that $\beta_1^{\theta=1} < 0$, and it follows that the second term of $D(x)$ for $x \leq \bar{x}^{\theta=1}$ is negative. Recall that $K^{\theta=1} = -q\left(1 - \frac{1}{\gamma}\right) = r + \frac{1}{\gamma}(\rho - r) + \frac{\gamma - 1}{2\gamma^2}\kappa^2$, and this leads to

$$r + \frac{\kappa^2}{\gamma} - K^{\theta=1} = -\frac{1}{\gamma}\left(\rho - r - \frac{\gamma + 1}{2\gamma}\kappa^2\right) < 0$$

when (5.2) holds. Since $A_1^{\theta=1} > 0$, the first term of $D(x)$ for $x \leq \bar{x}^{\theta=1}$ is also negative from (5.2) and thus $D(x) < 0$ for $x \leq \bar{x}^{\theta=1}$.

For $x \geq \bar{x}^{\theta=1}$,

$$r + \frac{\kappa^2}{\gamma_2} - K_2^{\theta=1} = -\frac{1}{\gamma_2}\left(\rho - r - \frac{\gamma_2 + 1}{2\gamma_2}\kappa^2\right) < 0$$

when (5.2) holds because $K_2^{\theta=1} = -q\left(1 - \frac{1}{\gamma_2}\right) = r + \frac{1}{\gamma_2}(\rho - r) + \frac{\gamma_2 - 1}{2\gamma_2^2}\kappa^2$. Since $A_2^{\theta=1} > 0$, the first term of $D(x)$ for $x \geq \bar{x}^{\theta=1}$ is negative.

Regarding the sign of the second term of $D(x)$ for $x \geq \bar{x}^{\theta=1}$, note that $n_+ > 1$ and $\alpha_2^{\theta=1} > 0$. However, the sign of $-r + \kappa^2(n_+ - 1)$ is unclear. If $-r + \kappa^2(n_+ - 1) < 0$, the second term of $D(x)$ for $x \geq \bar{x}^{\theta=1}$ is also negative and thus $D(x) < 0$ for $x \geq \bar{x}^{\theta=1}$. If $-r + \kappa^2(n_+ - 1) \geq 0$, let us consider $D'(x)$ for $x > \bar{x}^{\theta=1}$, which is given as

$$\begin{aligned} D'(x) = \Bigg[& -\frac{1}{\gamma_2}\left(r + \frac{\kappa^2}{\gamma_2} - K_2^{\theta=1}\right) A_2^{\theta=1} \mathcal{Z}_2^{\theta=1}(x)^{-\frac{1}{\gamma_2} - 1} \\ & + \left\{-r + \kappa^2(n_+ - 1)\right\} n_+ \alpha_2^{\theta=1}(n_+ - 1)\mathcal{Z}_2^{\theta=1}(x)^{n_+ - 2}\Bigg] \frac{d\mathcal{Z}_2^{\theta=1}(x)}{dx}, \end{aligned} \tag{B.4}$$

for $x > \bar{x}^{\theta=1}$. Since $\frac{d\mathcal{Z}_2^{\theta=1}(x)}{dx} < 0$, we can deduce from (B.4) that $D'(x) < 0$ for $x > \bar{x}^{\theta=1}$ if $-r + \kappa^2(n_+ - 1) \geq 0$. Since we have shown that $D(x) < 0$ for $x \leq \bar{x}^{\theta=1}$, it is clear that $D(\bar{x}^{\theta=1}) < 0$. Combining $D(\bar{x}^{\theta=1}) < 0$ and $D'(x) < 0$ for $x > \bar{x}^{\theta=1}$, we have $D(x) < 0$ for $x \geq \bar{x}^{\theta=1}$ even if $-r + \kappa^2(n_+ - 1) \geq 0$, and we can conclude that $D(x) < 0$ for $x \geq \bar{x}^{\theta=1}$ regardless of the sign of $-r + \kappa^2(n_+ - 1)$.

In summary, we have shown that $D(x) < 0$ for all x when (5.2) holds, and this completes the proof.

C Numerical Scheme

The numerical scheme presented in this section is the cornerstone of the analysis when investigating the implications studied in Section 4.

To set up some boundary values in the numerical scheme, we need the following lemma.

Lemma 1. *The following results hold:*

$$\lim_{z\to\infty} z^{-1}\varphi_1(z) = \frac{w\bar{L}}{r}, \quad \lim_{z\to\infty} \varphi_1'(z) = \frac{w\bar{L}}{r}, \quad \lim_{z\to\infty} z\varphi_1''(z) = 0,$$
$$\lim_{z\to 0+} z^{\frac{1}{\gamma_2}-1}\varphi_2(z) = \frac{\gamma_2}{1-\gamma_2}A_2, \quad \lim_{z\to 0+} z^{\frac{1}{\gamma_2}}\varphi_2'(z) = -A_2, \quad \lim_{z\to 0+} z^{\frac{1}{\gamma_2}+1}\varphi_2''(z) = \frac{1}{\gamma_2}A_2.$$

Proof. Proof. Using L'Hôpital's rule, we can show that

$$F_1(\infty) \triangleq \lim_{y\to\infty} F_1(y) = A_1, \quad F_2(0+) \triangleq \lim_{y\to 0+} F_2(y) = \frac{A_2}{\eta}. \tag{C.1}$$

By computing the derivatives of $\varphi_i(z)$ and applying (C.1), we can obtain the limits. □

As mentioned in Section 3, we solve the integral equations for $F_1(y)$, $F_2(y)$, and $F_1(1)$ instead of solving the integral equations for $\varphi_1(z)$, $\varphi_2(z)$, and $\bar{z}$. This approach has advantages thanks to the following two reasons. First, the domains of $F_1(y)$ and $F_2(y)$ are fixed intervals $[1,\infty)$ and $(0,1]$, respectively, without a free boundary, whereas the domains of $\varphi_1(z)$ and $\varphi_2(z)$ include the free boundary $\bar{z}$, an unknown that should be determined simultaneously when solving the equations for $\varphi_1(z)$ and $\varphi_2(z)$. In fact, in our approach, the role of $\bar{z}$ as an unknown variable is replaced by $F_1(1)$, which is a function value at a fixed point. Thus, we can obtain the numerical solutions to the functions $F_1(y)$ and $F_2(y)$ without considering the free boundary $\bar{z}$. After we get $F_1(1)$, the free boundary $\bar{z}$ is determined explicitly as (A.6). The other advantage is that the integral equations for $F_1(y)$, $F_2(y)$, and $F_1(1)$ do not include derivatives of $F_1(y)$ and $F_2(y)$ (see (A.26), (A.27), and (A.28)). In contrast, to solve the integral equations for $\varphi_1(z)$, $\varphi_2(z)$, and $\bar{z}$, we need to compute the derivatives of the unknown functions $\varphi_1(z)$ and $\varphi_2(z)$ because $\varphi_i(\cdot)$ always appears in $\alpha_i(z)$ and $\beta_i(z)$ as $\varphi_i(z) - z\varphi_i'(z)$ for $i = 1, 2$.

Overall, it is much easier to develop a numerical scheme to solve the integral equations for $F_1(y)$, $F_2(y)$, and $F_1(1)$ in fixed domains without a free boundary that do not require the computation of derivatives of the unknown functions.

Note that the domain of $F_2(y)$ is $(0,1]$. We discretize the domain of $F_2(y)$ using $y_{2,k}^{(N)} \triangleq \frac{k}{N}$ for $k=0,1,\ldots,N$, and consider the following simple function $F_2^{(N)}(y)$:

$$F_2^{(N)}(y) = F_{2,0}^{(N)}\mathbf{1}_{\{0<y\le\frac{1}{2N}\}} + \sum_{k=1}^{N-1} F_{2,k}^{(N)}\mathbf{1}_{\{y_{2,k}^{(N)}-\frac{1}{2N}<y\le y_{2,k}^{(N)}+\frac{1}{2N}\}} + F_{2,N}^{(N)}\mathbf{1}_{\{1-\frac{1}{2N}<y\le 1\}}, \tag{C.2}$$

$F_{2,0}^{(N)}$ and $F_{2,N}^{(N)}$ correspond to $F_2(0+)$ and $F_2(1)$, respectively, and we set

$$F_{2,0}^{(N)} = \frac{A_2}{\eta} \tag{C.3}$$

by (C.1). Similarly, we introduce a simple function $F_1^{(M)}(y)$ that approximates $F_1(y)$. However, since the domain of $F_1(y)$ is $[0,\infty)$, which is unbounded, additional treatment is necessary. Note that $F_1(\infty)=A_1$ in (C.1). Thus, we set $F_1^{(M)}(y)=A_1$ for $y\ge y_{max}$ for some large enough y_{max} and discretize the interval $[1, y_{max}]$ into M sub-intervals. To do so, let us define $\Delta y_1^{(M)} \triangleq \frac{y_{max}-1}{M}$, $\quad y_{1,j}^{(M)} \triangleq 1 + j\Delta y_1^{(M)}$ for $j=0,1,\ldots,M$. Then, the simple function $F_1^{(M)}(y)$ is defined as follows:

$$F_1^{(M)}(y) = F_{1,0}^{(M)}\mathbf{1}_{\{1\le y\le 1+\frac{1}{2}\Delta y_1^{(M)}\}} + \sum_{j=1}^{M-1} F_{1,j}^{(M)}\mathbf{1}_{\{y_{1,j}^{(M)}-\frac{1}{2}\Delta y_1^{(M)}<y\le y_{1,j}^{(M)}+\frac{1}{2}\Delta y_1^{(M)}\}} + F_{1,M}^{(M)}\mathbf{1}_{\{y_{max}-\frac{1}{2}\Delta y_1^{(M)}<y\}}. \tag{C.4}$$

Here, $F_{1,0}^{(M)}$ corresponds to $F_1(1)$ satisfying (A.28), and we set

$$F_{1,M}^{(M)} = A_1. \tag{C.5}$$

If we replace $F_2(\cdot)$ in (A.27) by the simple function $F_2^{(N)}(\cdot)$ in (C.2) that approximates $F_2(\cdot)$

and set $y = y_{2,k}^{(N)}$ for $k = 1, \ldots, N-1$, we can derive the following matrix equation:

$$\begin{bmatrix} F_{2,1}^{(N)} \\ F_{2,2}^{(N)} \\ \vdots \\ F_{2,N-1}^{(N)} \end{bmatrix} = \mathbf{M_2} \begin{bmatrix} F_{2,1}^{(N)^{-\frac{\theta-1}{\gamma_1}}} \\ F_{2,2}^{(N)^{-\frac{\theta-1}{\gamma_1}}} \\ \vdots \\ F_{2,N-1}^{(N)^{-\frac{\theta-1}{\gamma_1}}} \end{bmatrix} + \mathbf{b_2}(F_{2,0}^{(N)}, F_{2,N}^{(N)}) \tag{C.6}$$

for an $(N-1) \times (N-1)$ matrix $\mathbf{M_2}$ and an $(N-1) \times 1$ vector $\mathbf{b_2}$ that depends on $F_{2,0}^{(N)}$ and $F_{2,N}^{(N)}$. Similarly, replacing $F_1(\cdot)$ in (A.26) by the simple function $F_1^{(M)}(\cdot)$ in (C.4) and setting $y = y_{1,j}^{(M)}$ for $j = 1, \ldots, M-1$, we have a matrix equation as follows:

$$\begin{bmatrix} F_{1,1}^{(M)} \\ F_{1,2}^{(M)} \\ \vdots \\ F_{1,M-1}^{(M)} \end{bmatrix} = \mathbf{M_1} \begin{bmatrix} F_{1,1}^{(M)^{-(\theta-1)\psi}} \\ F_{1,2}^{(M)^{-(\theta-1)\psi}} \\ \vdots \\ F_{1,M-1}^{(M)^{-(\theta-1)\psi}} \end{bmatrix} + \mathbf{b_1}(F_{1,0}^{(M)}, F_{1,M}^{(M)}) \tag{C.7}$$

for an $(M-1) \times (M-1)$ matrix $\mathbf{M_1}$ and an $(M-1) \times 1$ vector $\mathbf{b_1}$ that depends on $F_{1,0}^{(M)}$ and $F_{1,M}^{(M)}$. By replacing $F_1(\cdot)$ and $F_2(\cdot)$ on the right-hand side of (A.28) by $F_1^{(M)}(\cdot)$ in (C.4) and $F_2^{(N)}(\cdot)$ in (C.2), respectively, we can represent $F_{1,0}^{(M)}$ as follows:

$$F_{1,0}^{(M)} = \mathbf{R_1} \begin{bmatrix} F_{1,0}^{(M)^{-(\theta-1)\psi}} \\ F_{1,1}^{(M)^{-(\theta-1)\psi}} \\ \vdots \\ F_{1,M}^{(M)^{-(\theta-1)\psi}} \end{bmatrix} + F_{1,0}^{(M)^{-\frac{(\gamma_2-\gamma)(\theta-1)\psi}{\gamma_2}}} \mathbf{R_2} \begin{bmatrix} F_{2,0}^{(N)^{-\frac{\theta-1}{\gamma_1}}} \\ F_{2,1}^{(N)^{-\frac{\theta-1}{\gamma_1}}} \\ \vdots \\ F_{2,N}^{(N)^{-\frac{\theta-1}{\gamma_1}}} \end{bmatrix} \tag{C.8}$$

for a $1 \times (M+1)$ vector $\mathbf{R_1}$ and a $1 \times (N+1)$ vector $\mathbf{R_2}$. From (A.29), we have

$$F_{2,N}^{(N)} = F_{1,0}^{(M)^{\frac{\eta\gamma-\eta\psi\gamma+\psi\gamma}{\gamma_2}}} \zeta^{\psi\gamma(\frac{1}{\gamma_2}-\frac{1}{\gamma})}. \tag{C.9}$$

Using an iterative method for large enough M and N, we obtain $F_1^{(M)}(y)$ and $F_2^{(N)}(y)$ that approximate $F_1(y)$ and $F_2(y)$ with $F_{1,0}^{(M)}, F_{1,1}^{(M)}, \cdots, F_{1,M}^{(M)}$ and $F_{2,0}^{(N)}, F_{2,1}^{(N)}, \cdots, F_{2,N}^{(N)}$

satisfying (C.3), (C.5), (C.6), (C.7), (C.8), and (C.9) simultaneously.[20] Once we get $F_1^{(M)}(y)$ and $F_2^{(N)}(y)$, we can compute $\bar{z}$, $\alpha_1(z)$, $\beta_1(z)$, $\alpha_2(z)$, and $\beta_2(z)$ in Proposition A.1 using $F_1^{(M)}(y)$ and $F_2^{(N)}(y)$ instead of $F_1(y)$ and $F_2(y)$, and the optimal strategies in Proposition 1 can be attained numerically.

[20]Since the entries of $\mathbf{M_1}$, $\mathbf{M_2}$, $\mathbf{R_1}$, $\mathbf{R_2}$, $\mathbf{b_1}$, and $\mathbf{b_2}$ are very complicated, they are omitted from the paper. They can be provided upon request.